Mathematical Induction

Part Two

Outline for Today

- ***Variations on Induction***
  - Starting later, taking different step sizes, and more!
- ***"Build Up" versus "Build Down"***
  - An inductive nuance that follows from our general proofwriting principles.
- ***Complete Induction***
  - When one assumption isn't enough!

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$$P(0) \text{ is true}$$

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

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New Stuff!

Variations on Induction: ***Starting Later***

Induction Starting at 0

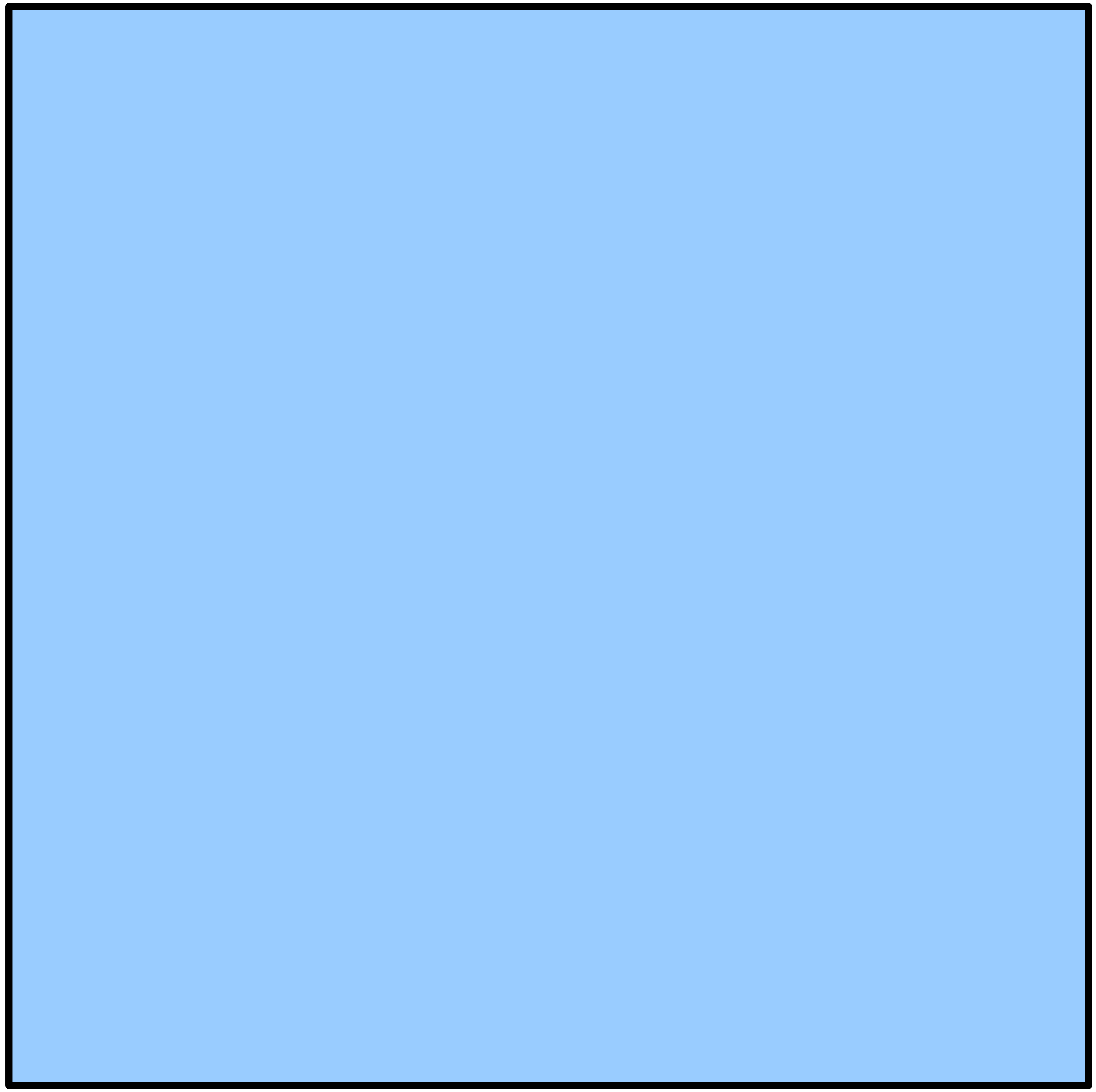
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: ***Bigger Steps***

Subdividing a Square



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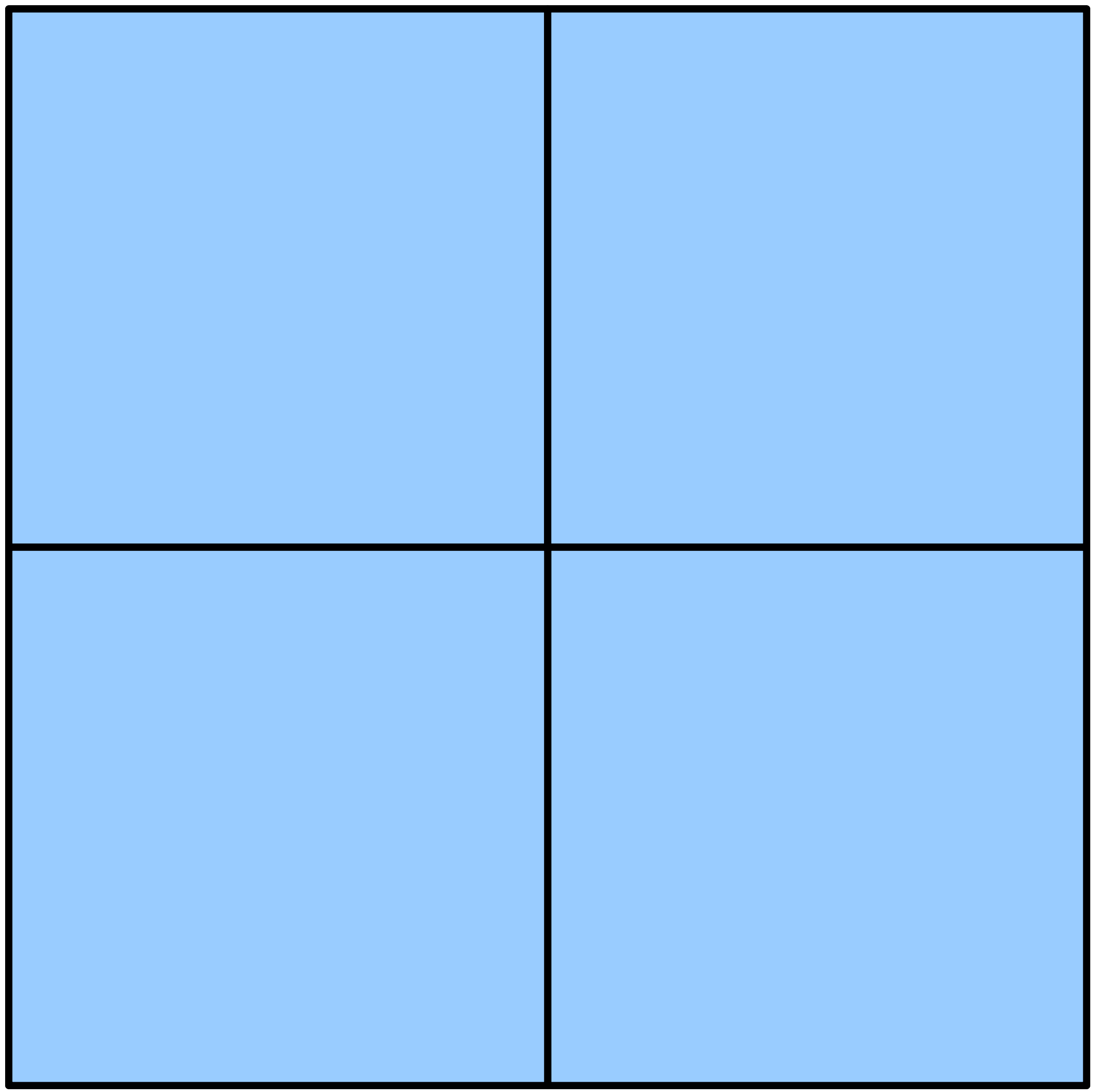

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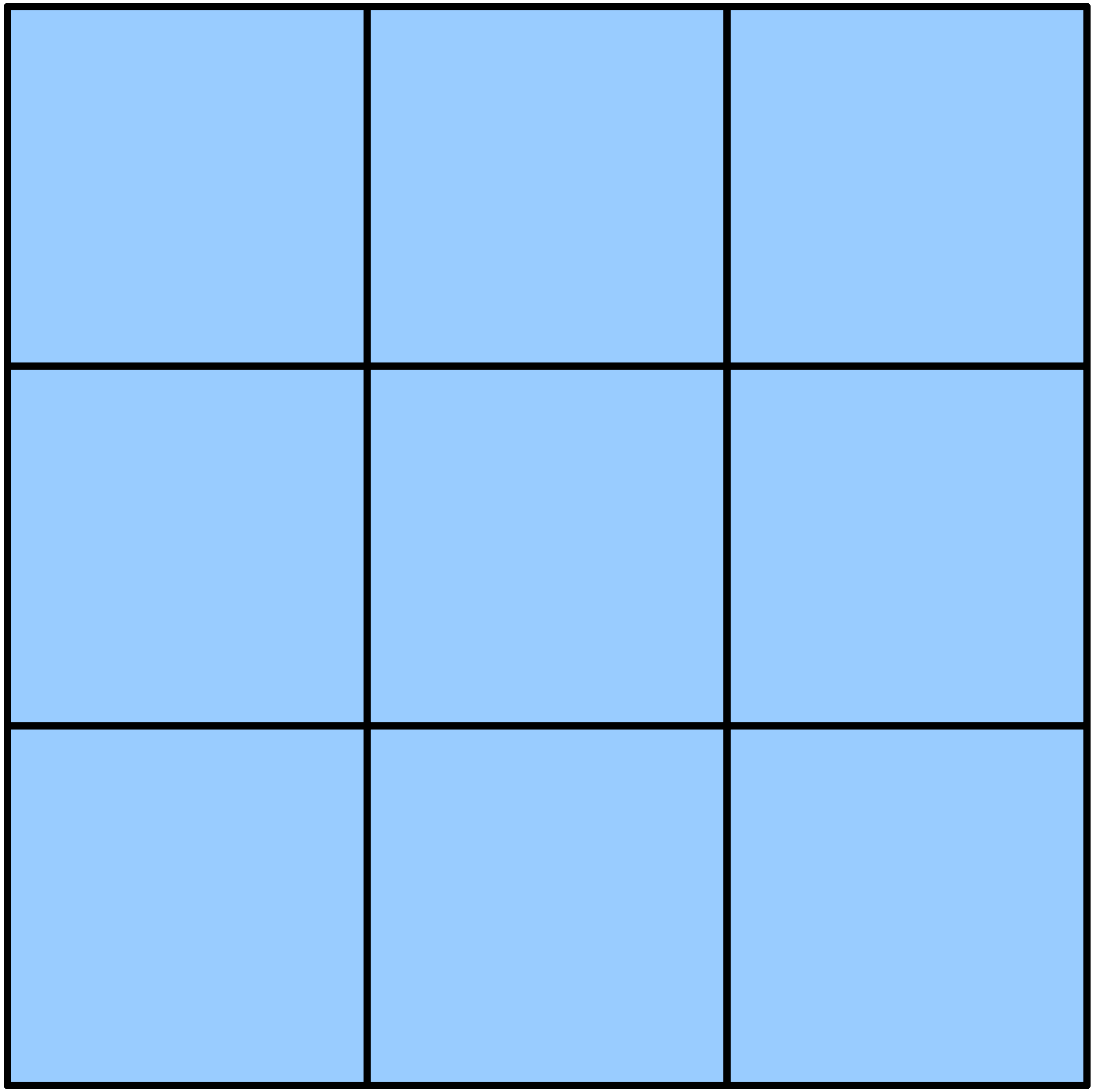

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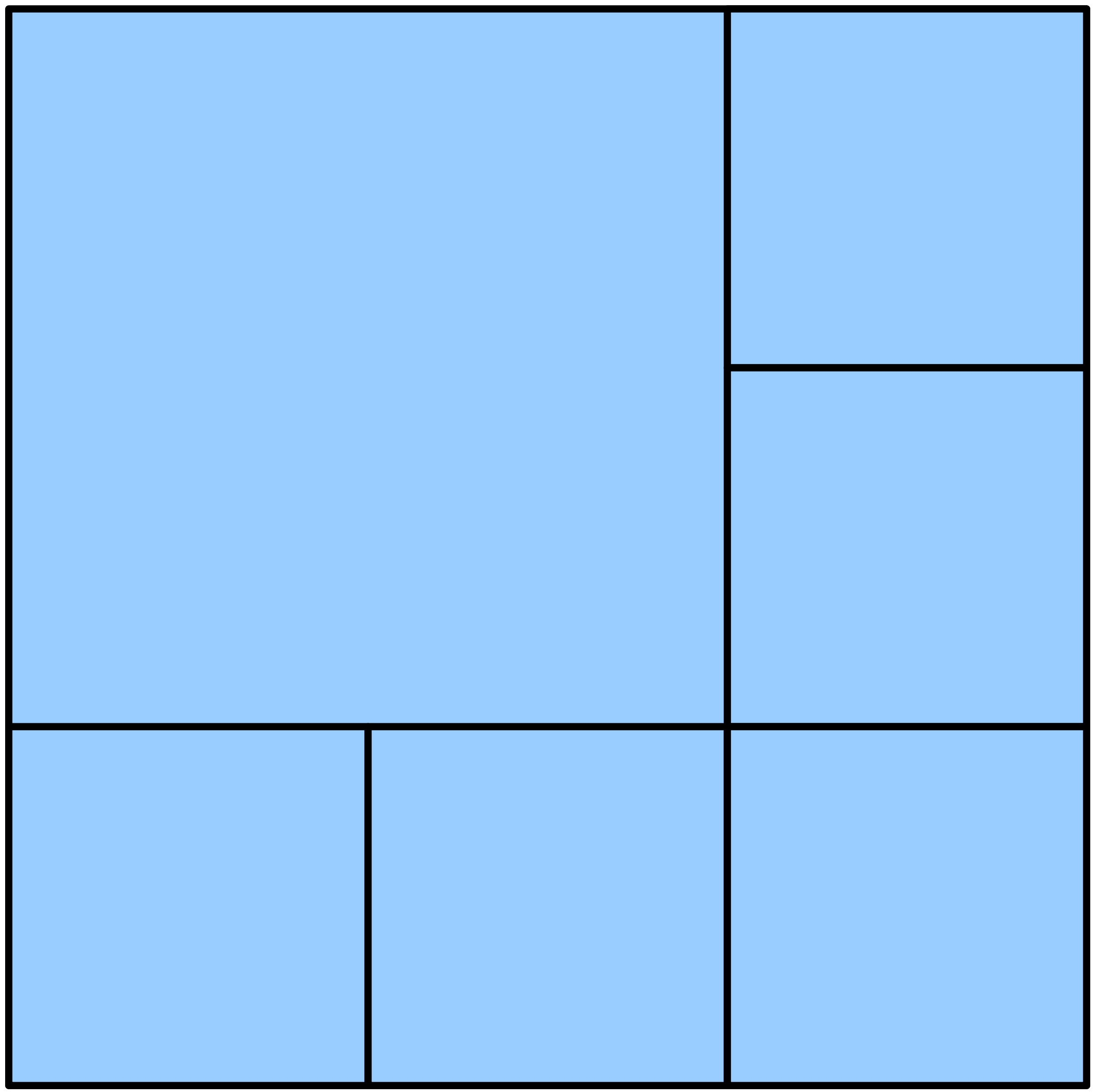

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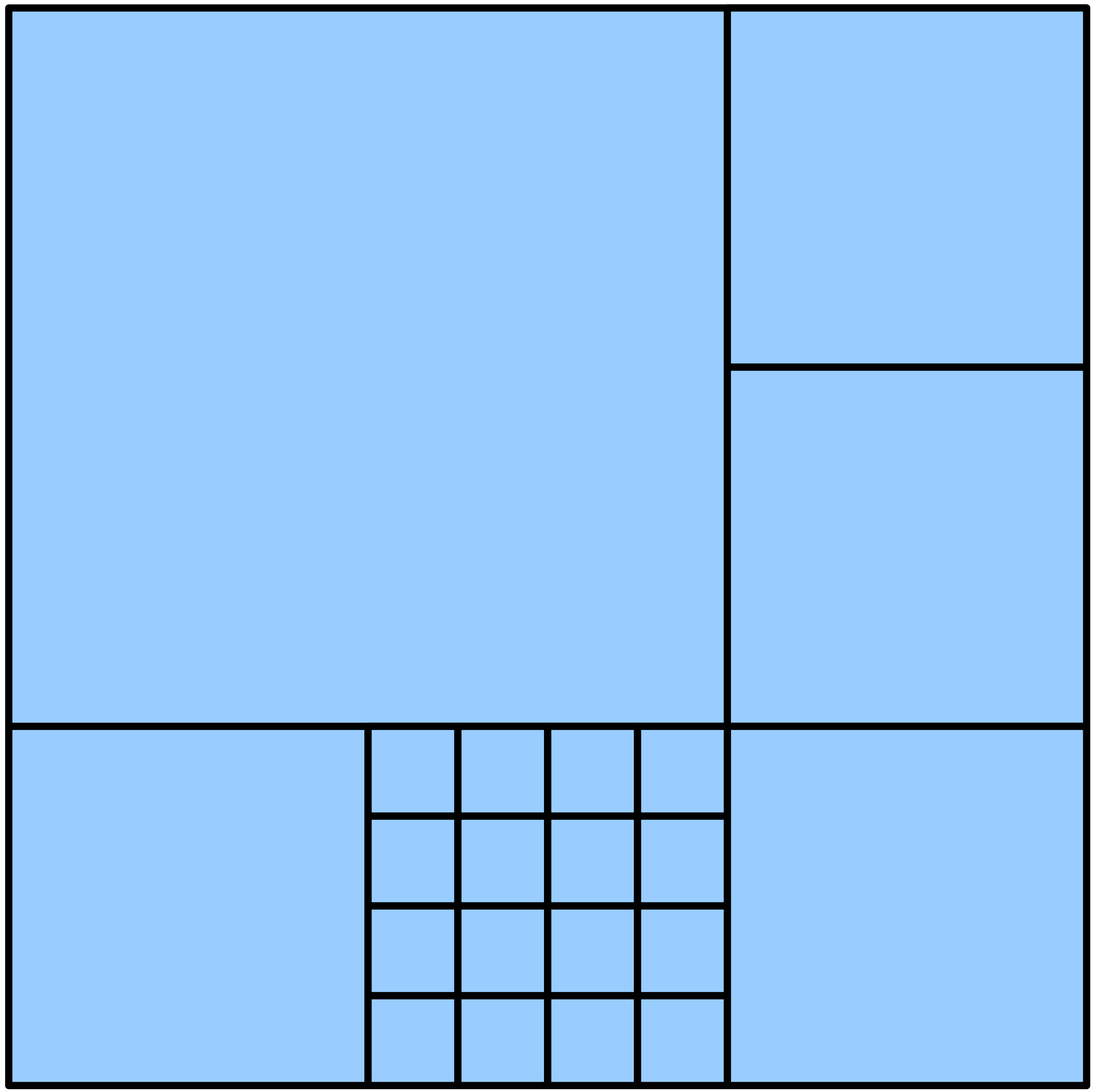

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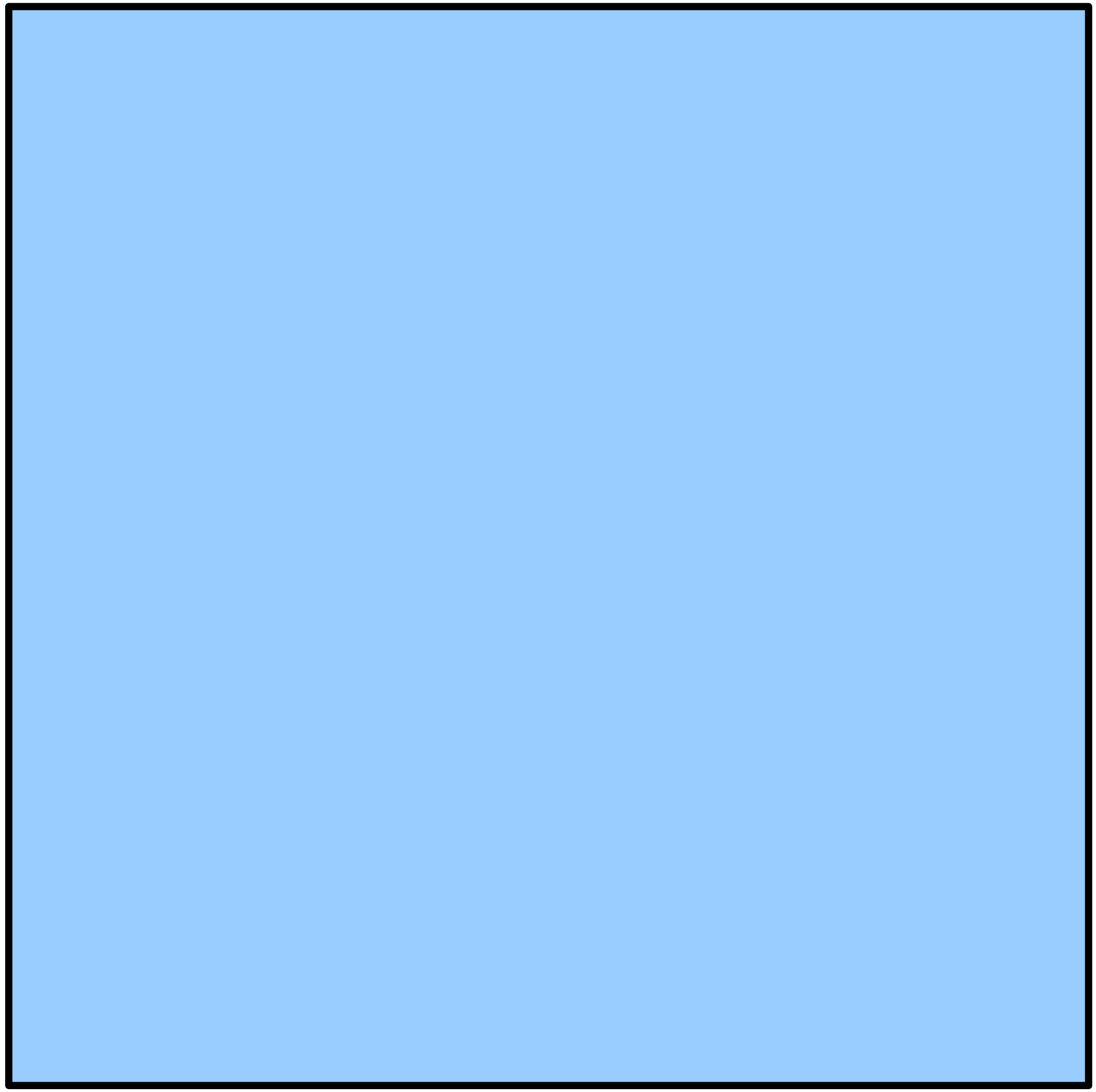

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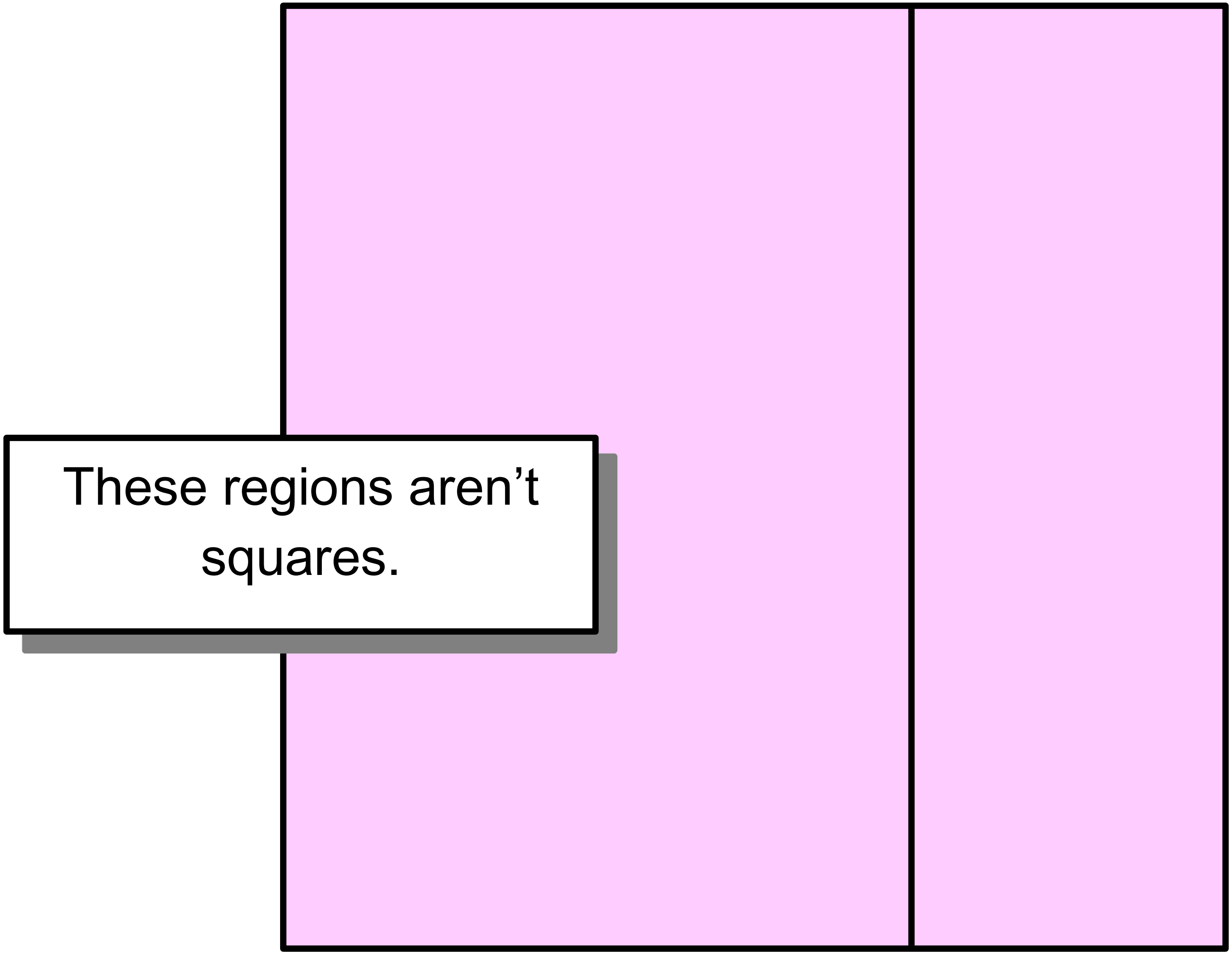

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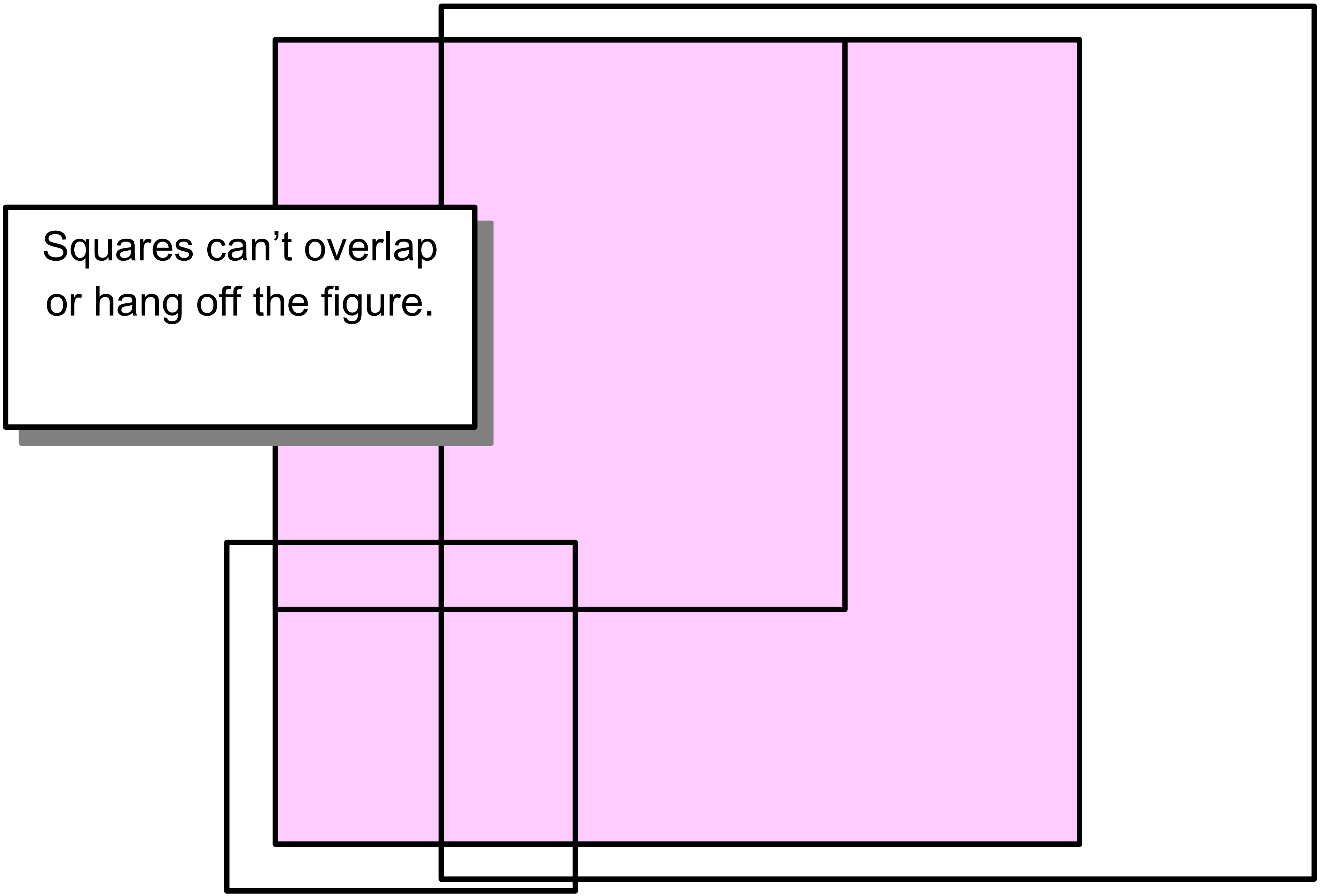

For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

Give it a try! Enter your guess as a list of values.

Respond at pollev.com/zhenglian740

1 2 3 4 5 6 7 8 9 10 11 12

1

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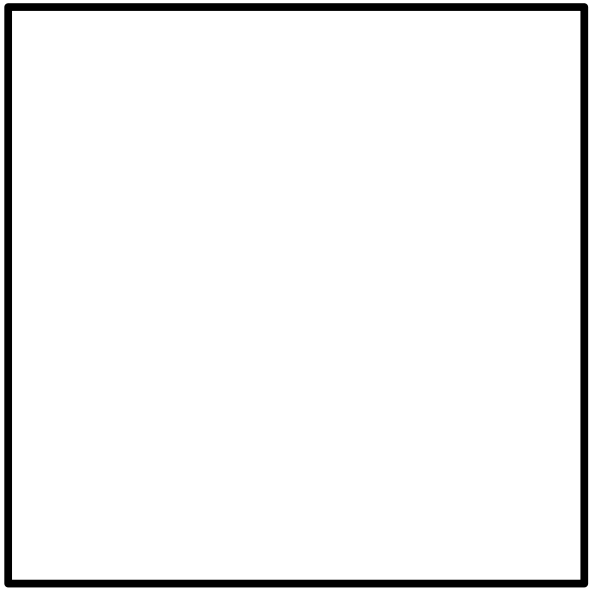
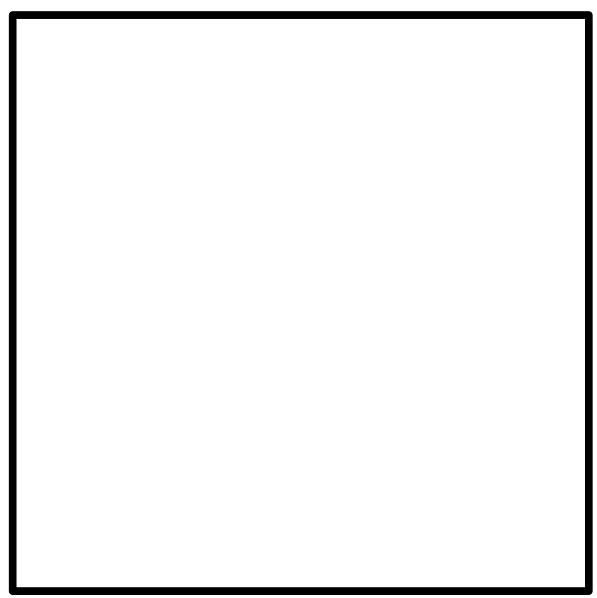


1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



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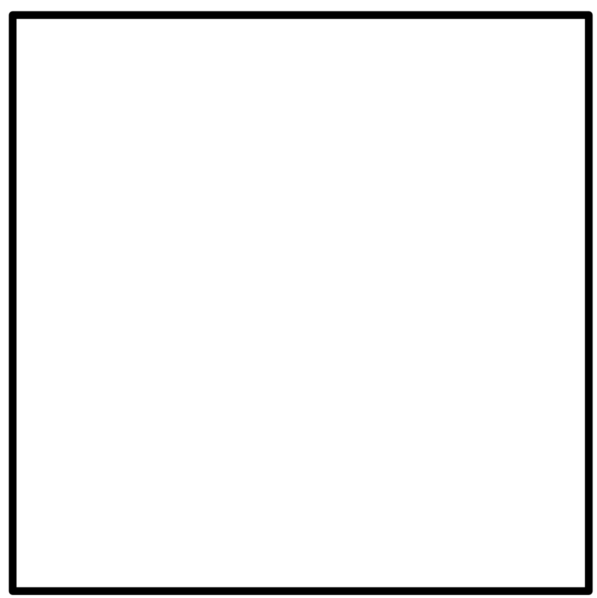
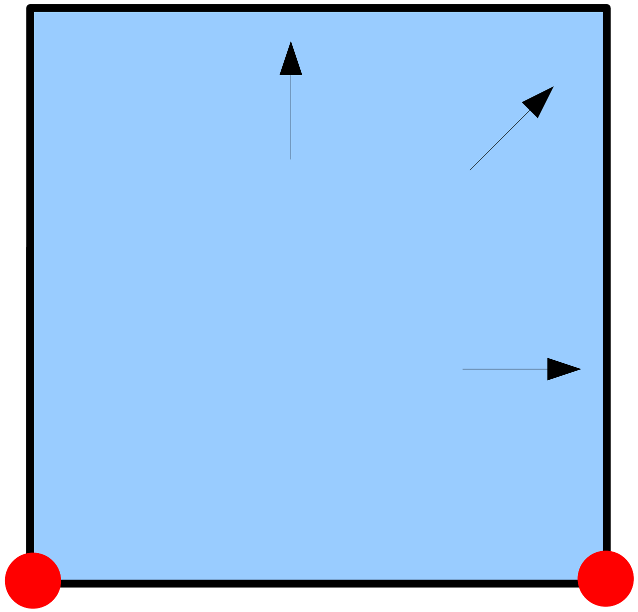


corners: 4

squares: <4

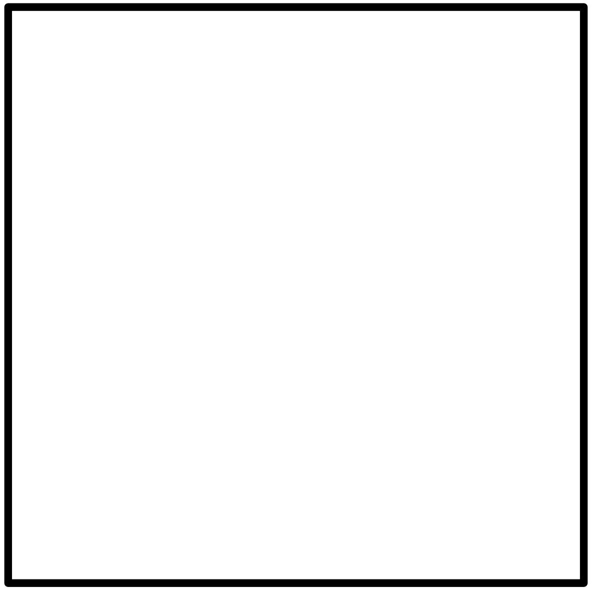
1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



By the pigeonhole principle, at least one smaller square needs to cover at least *two* of the original square's corners.

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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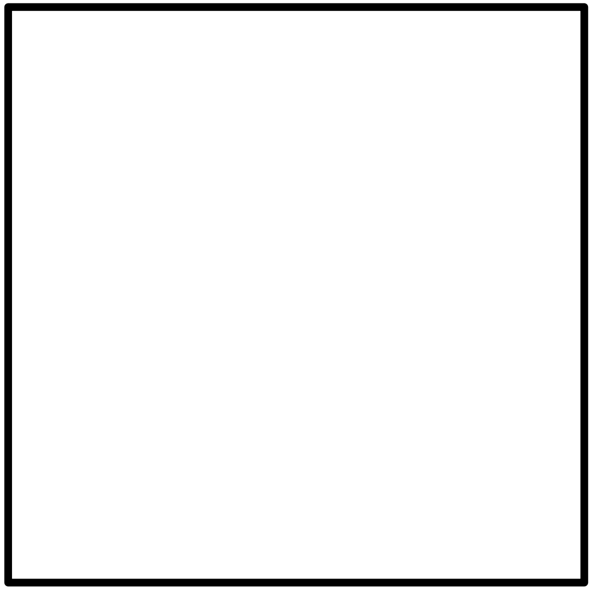
1	2
4	3

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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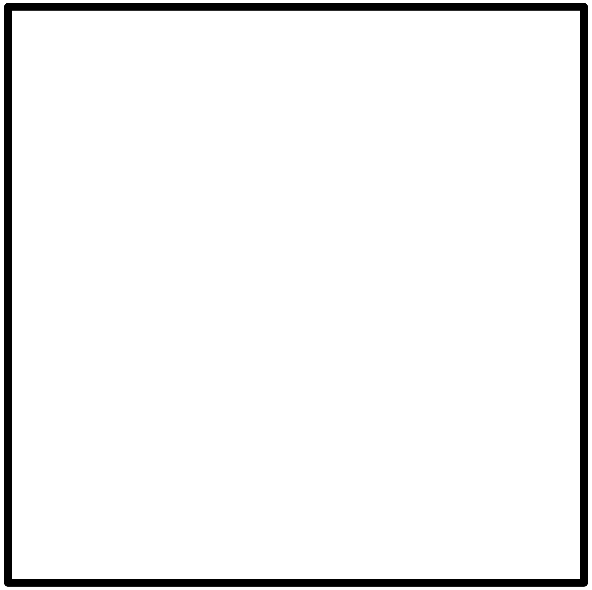
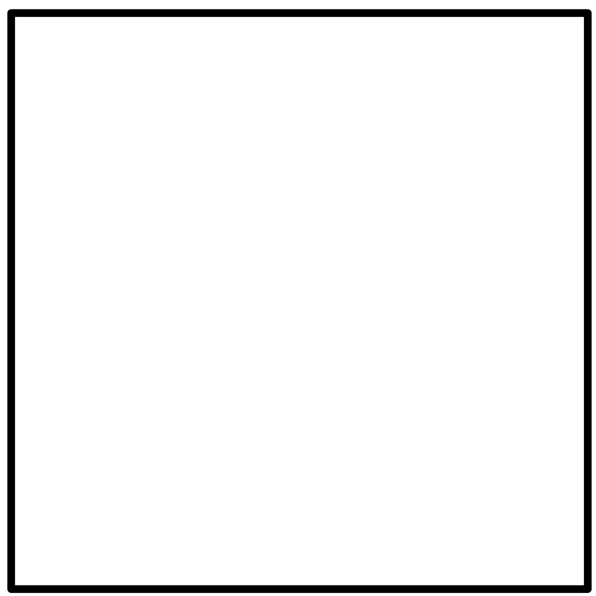


corners: 4

squares: 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

At least one square cannot be covering *any* of the original corners



corners: 4

squares: 5

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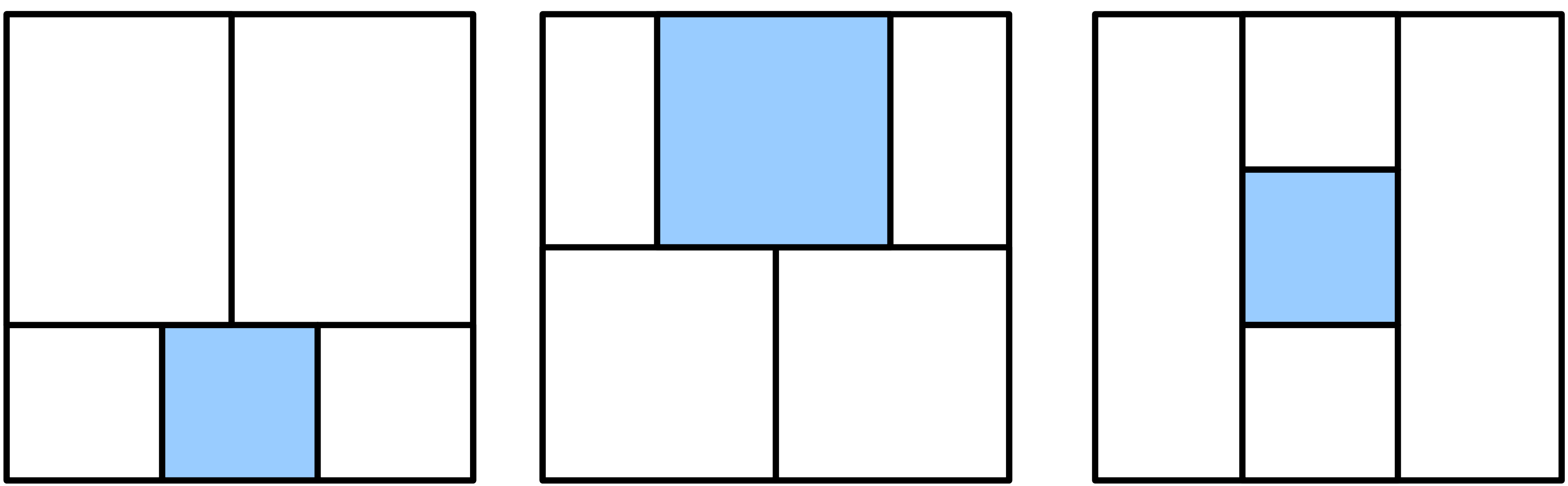

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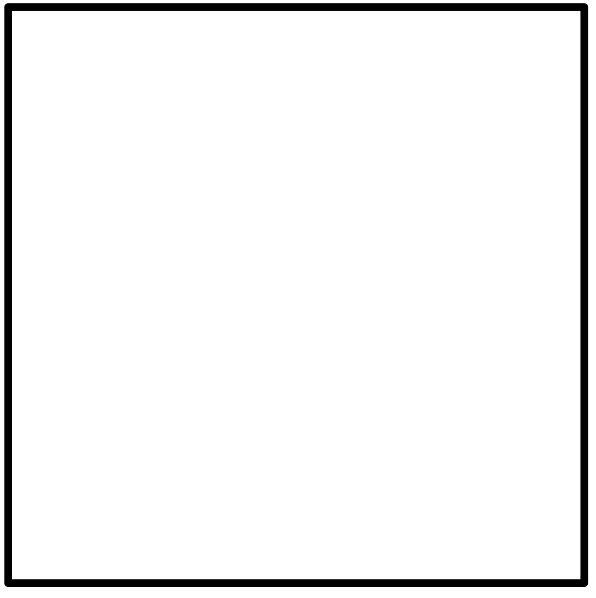

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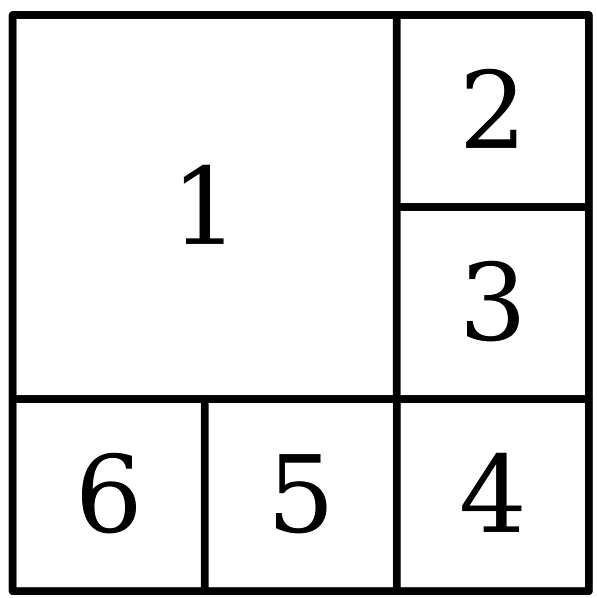

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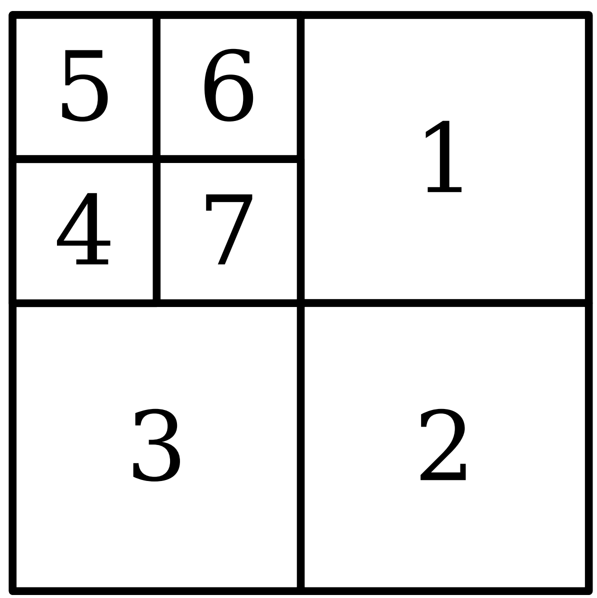

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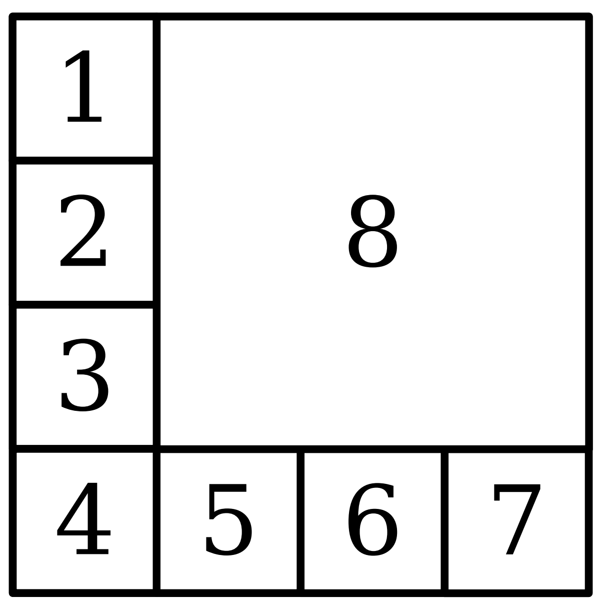

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

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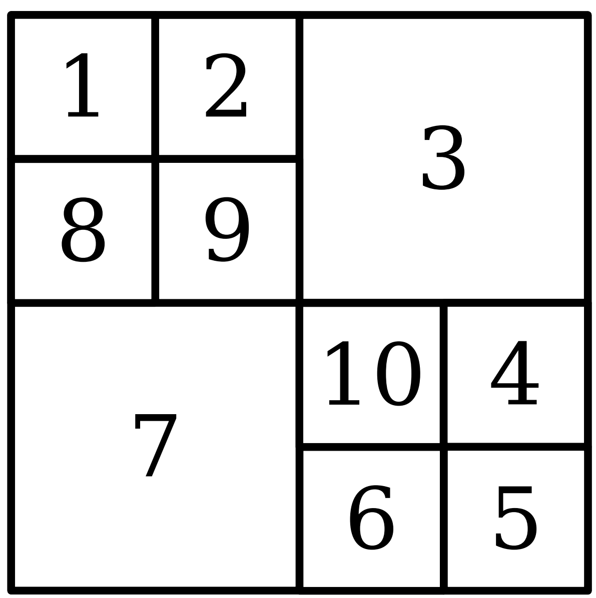

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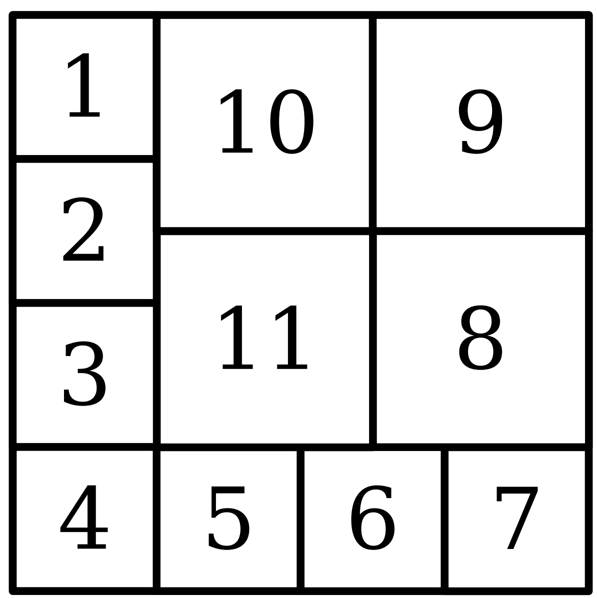

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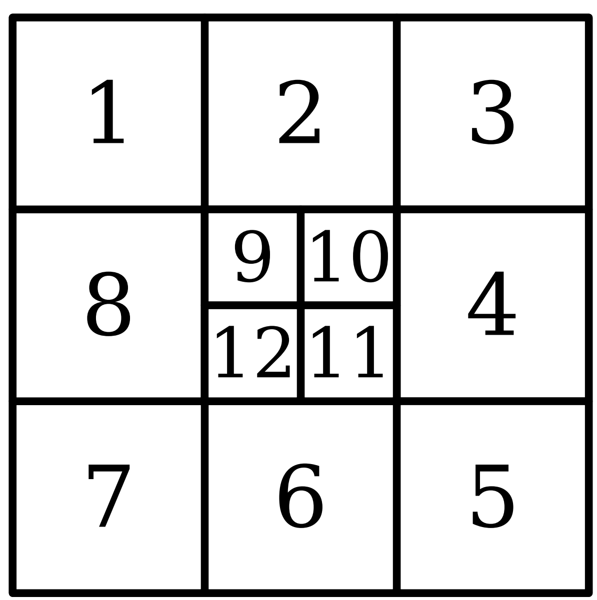

An Insight

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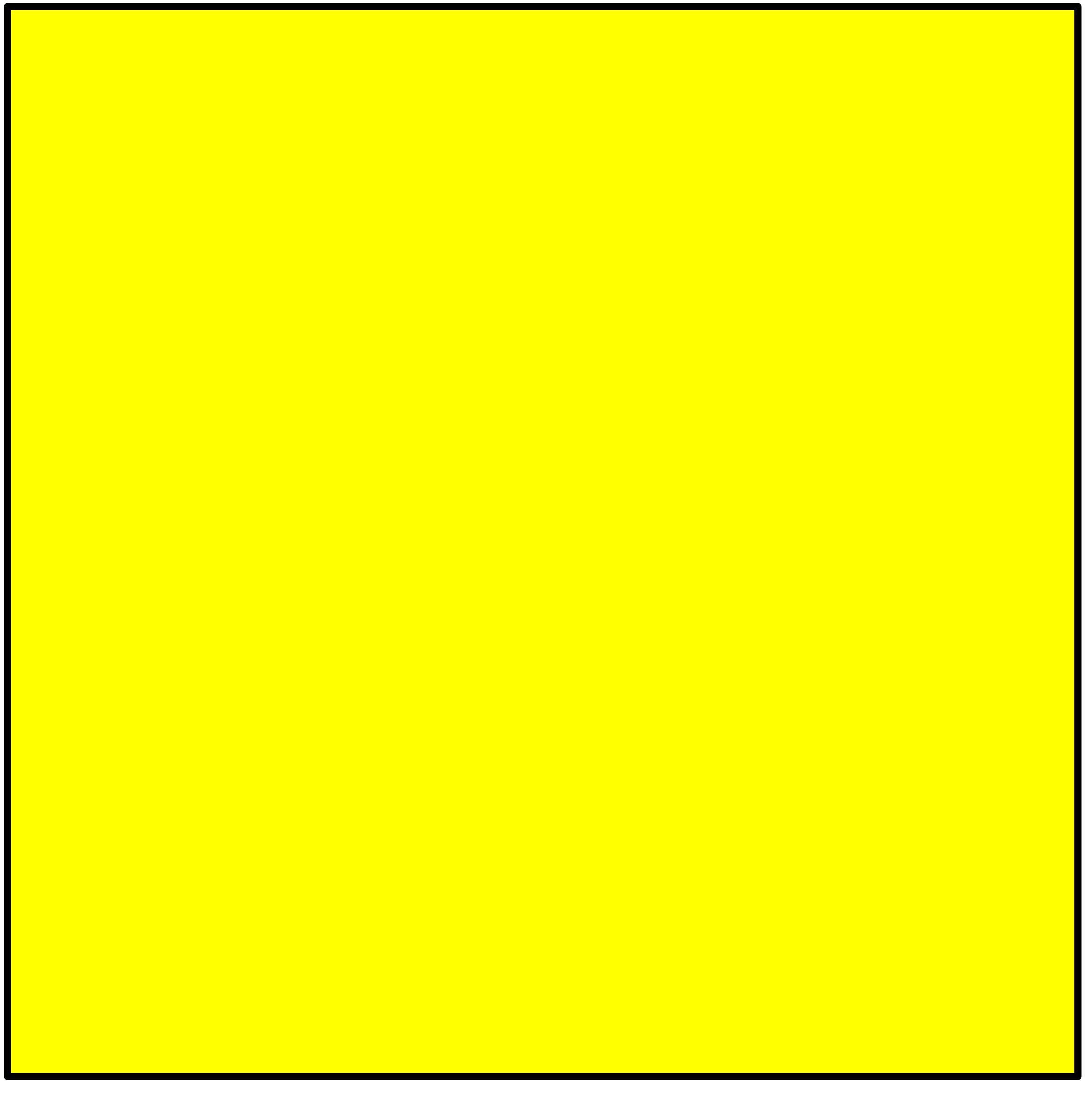

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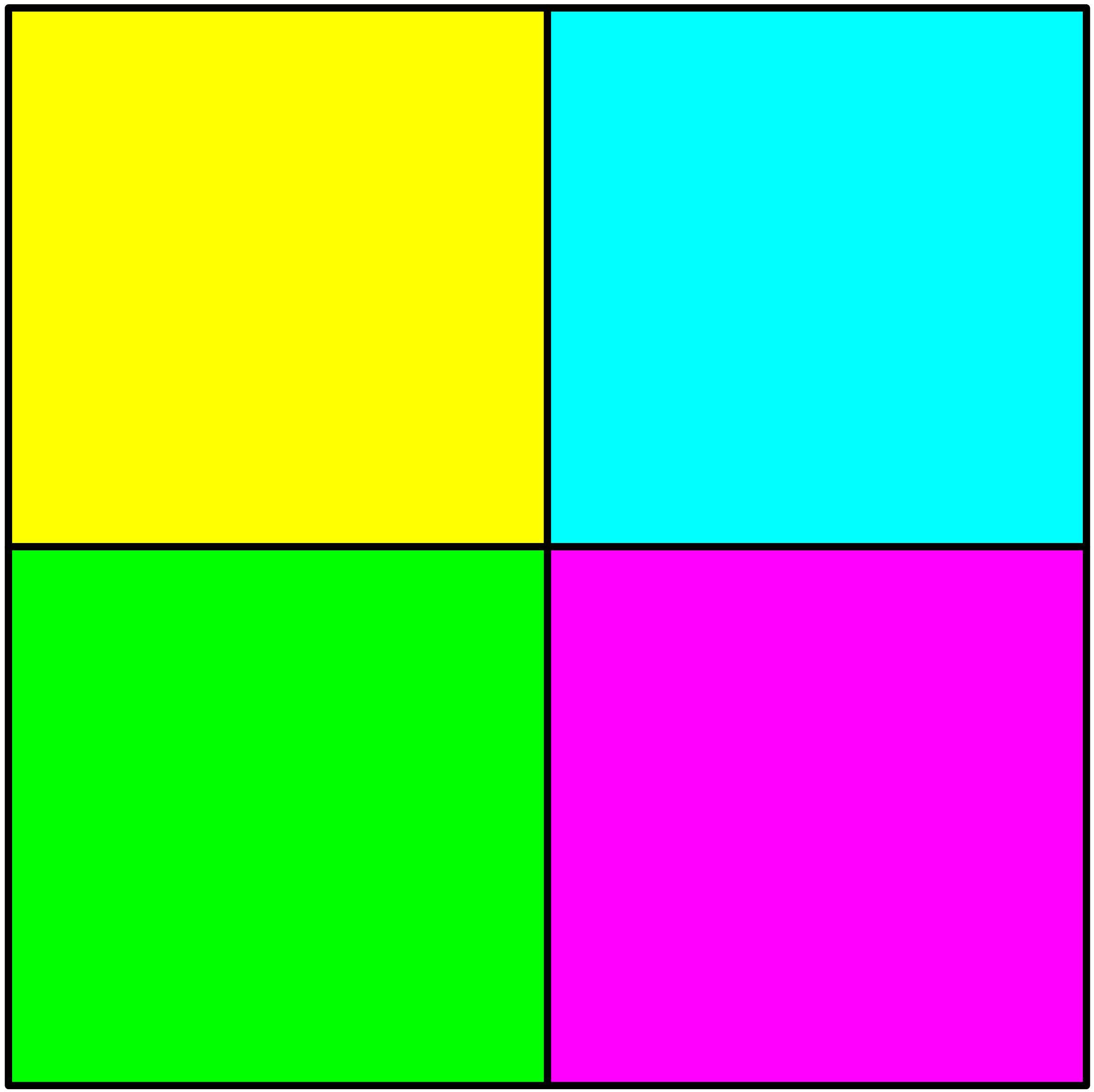

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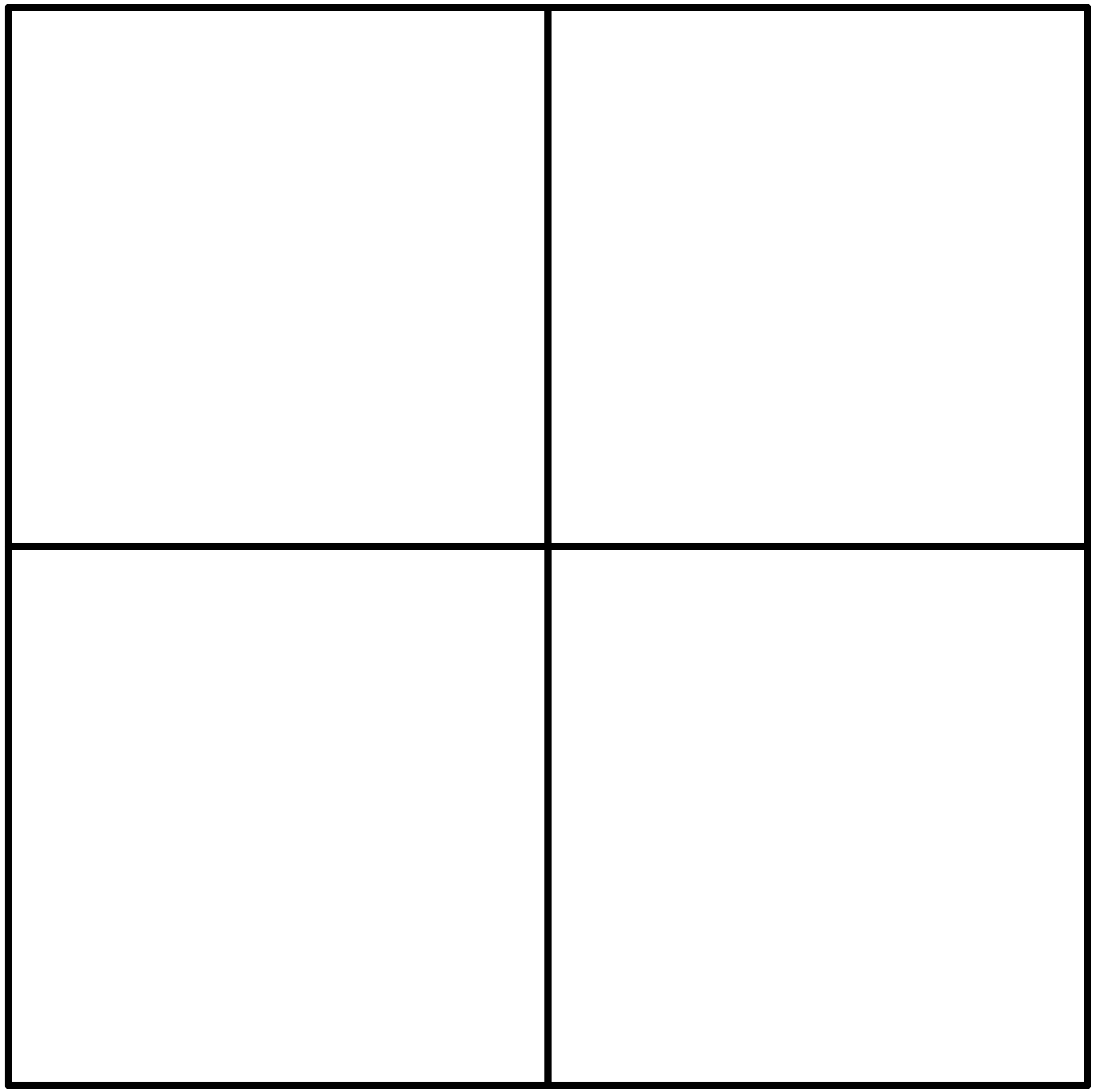

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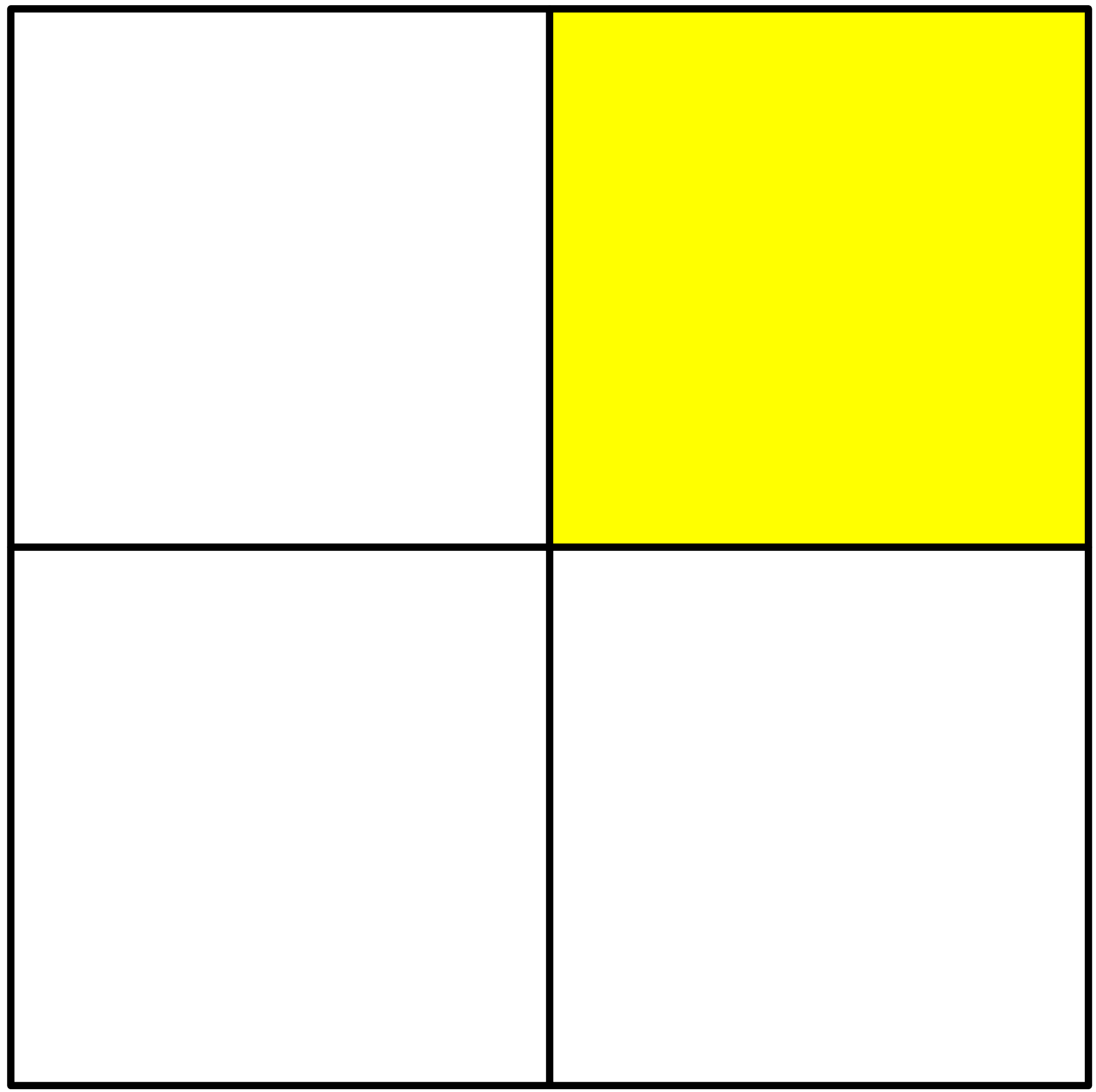

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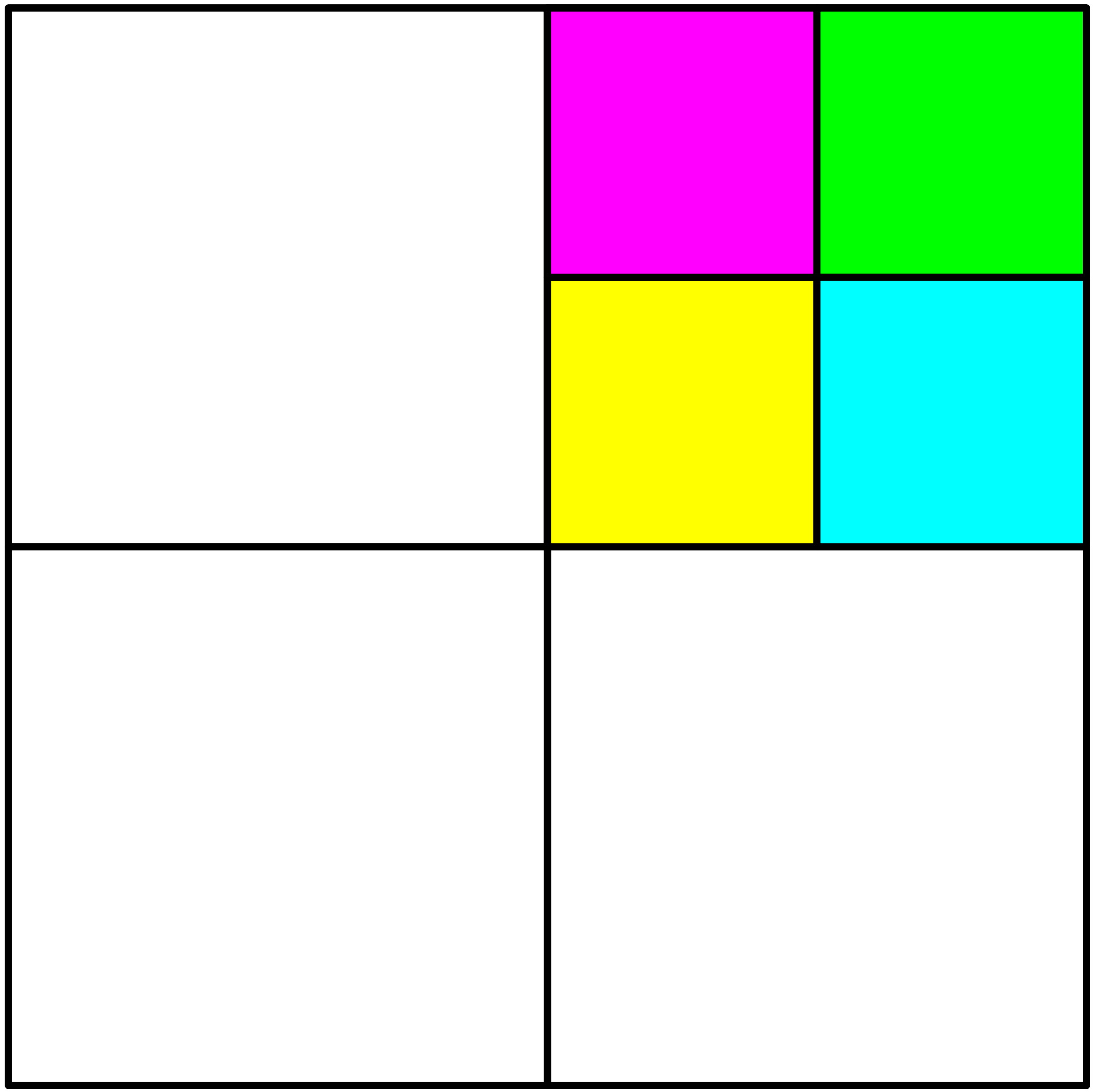

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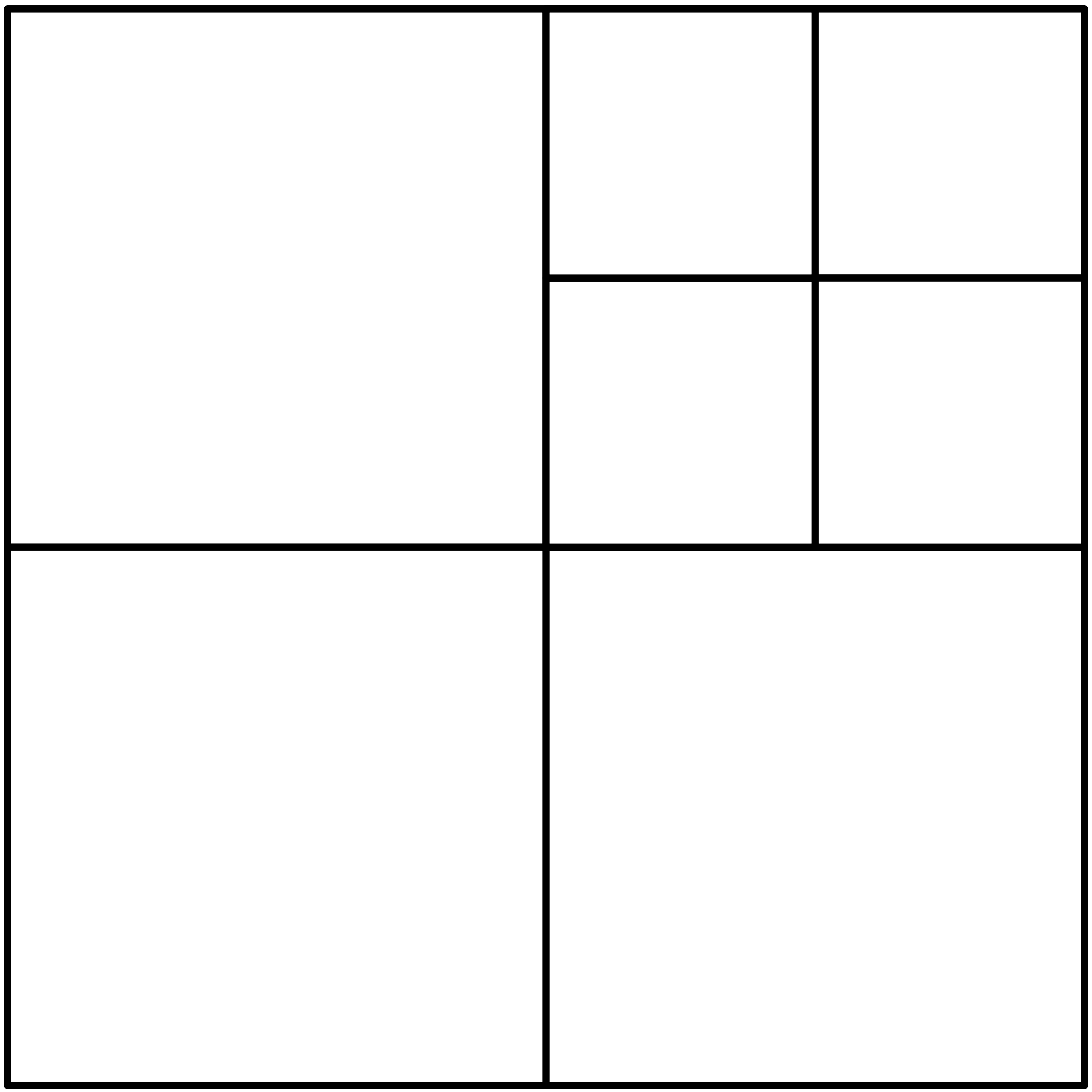

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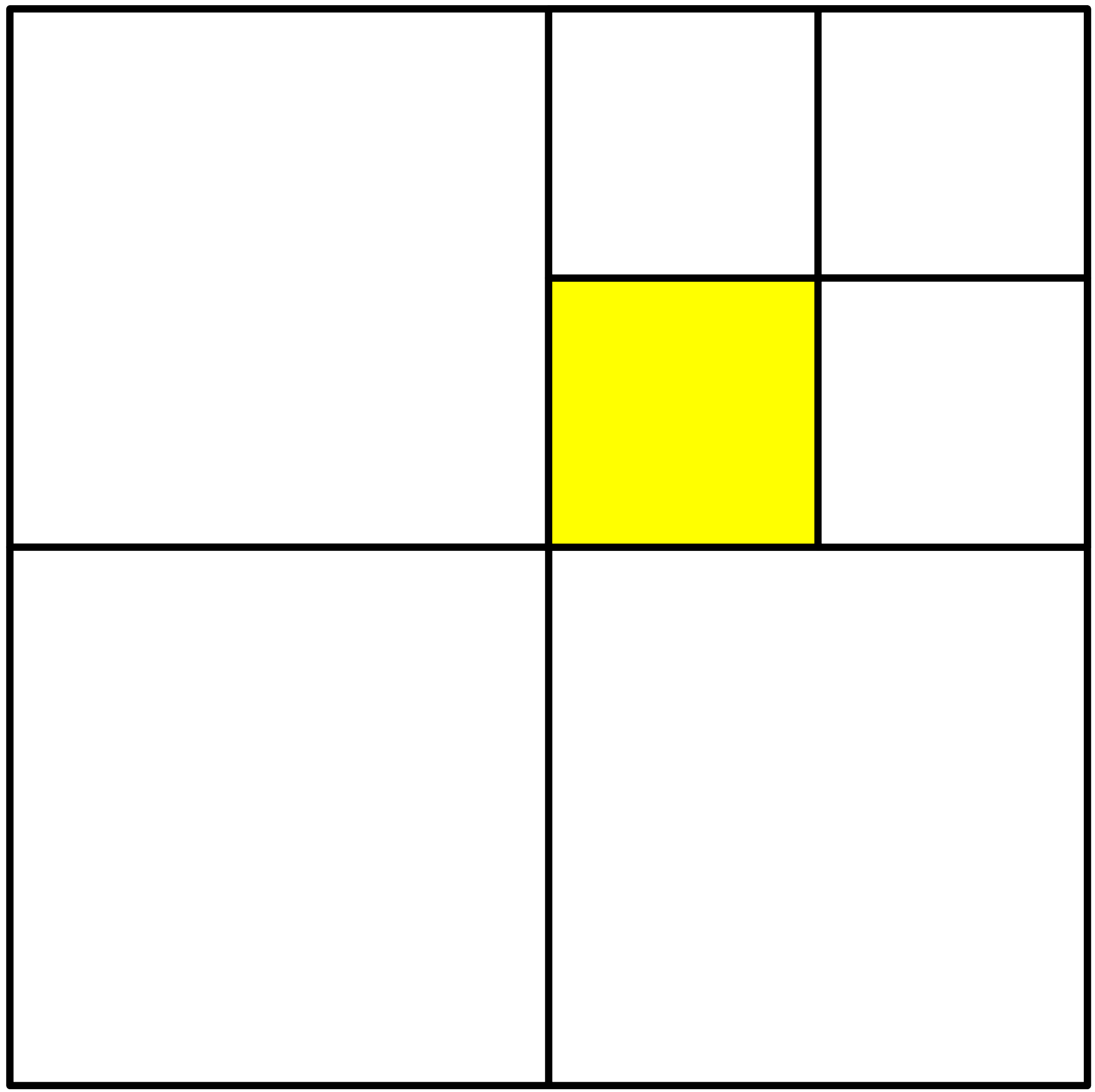

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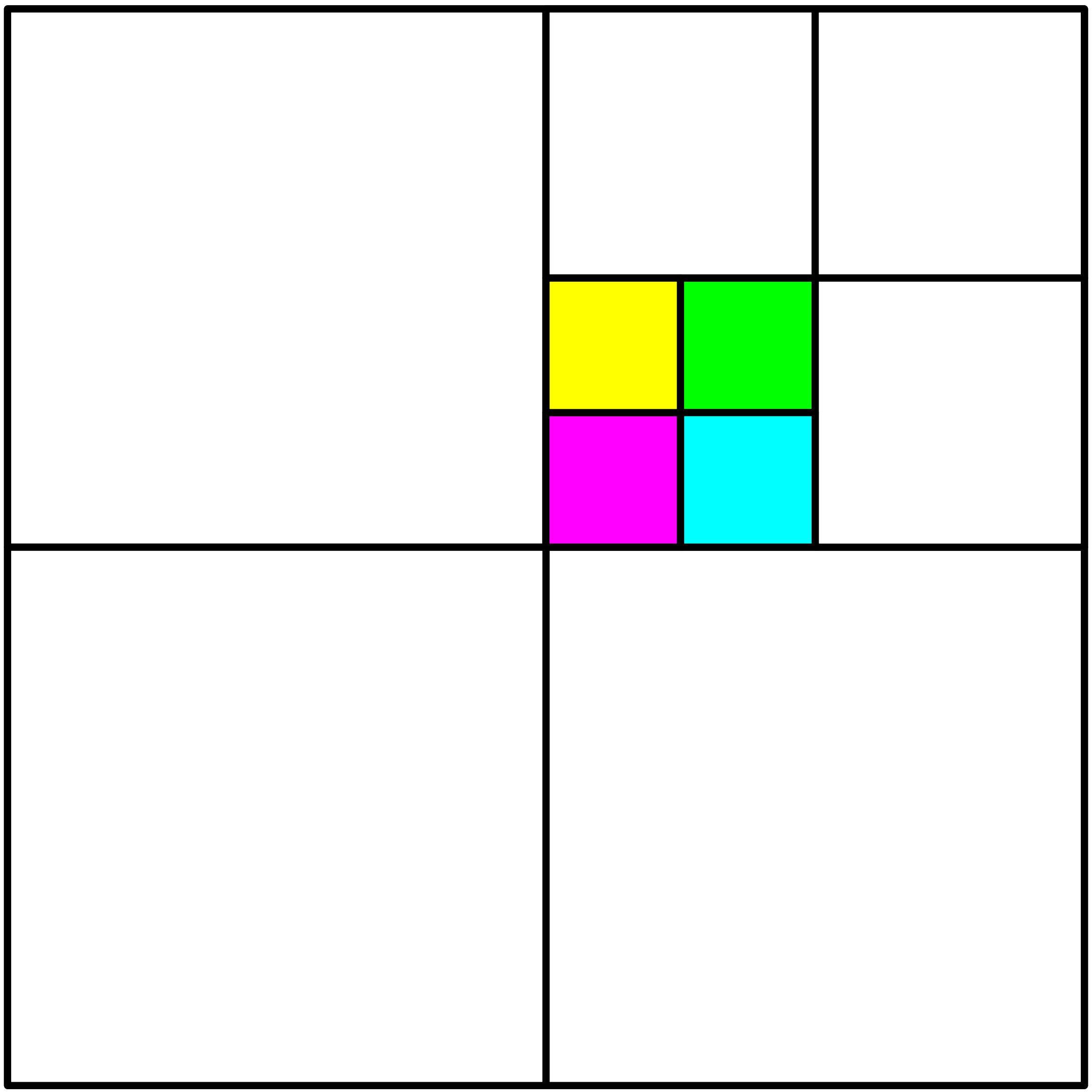

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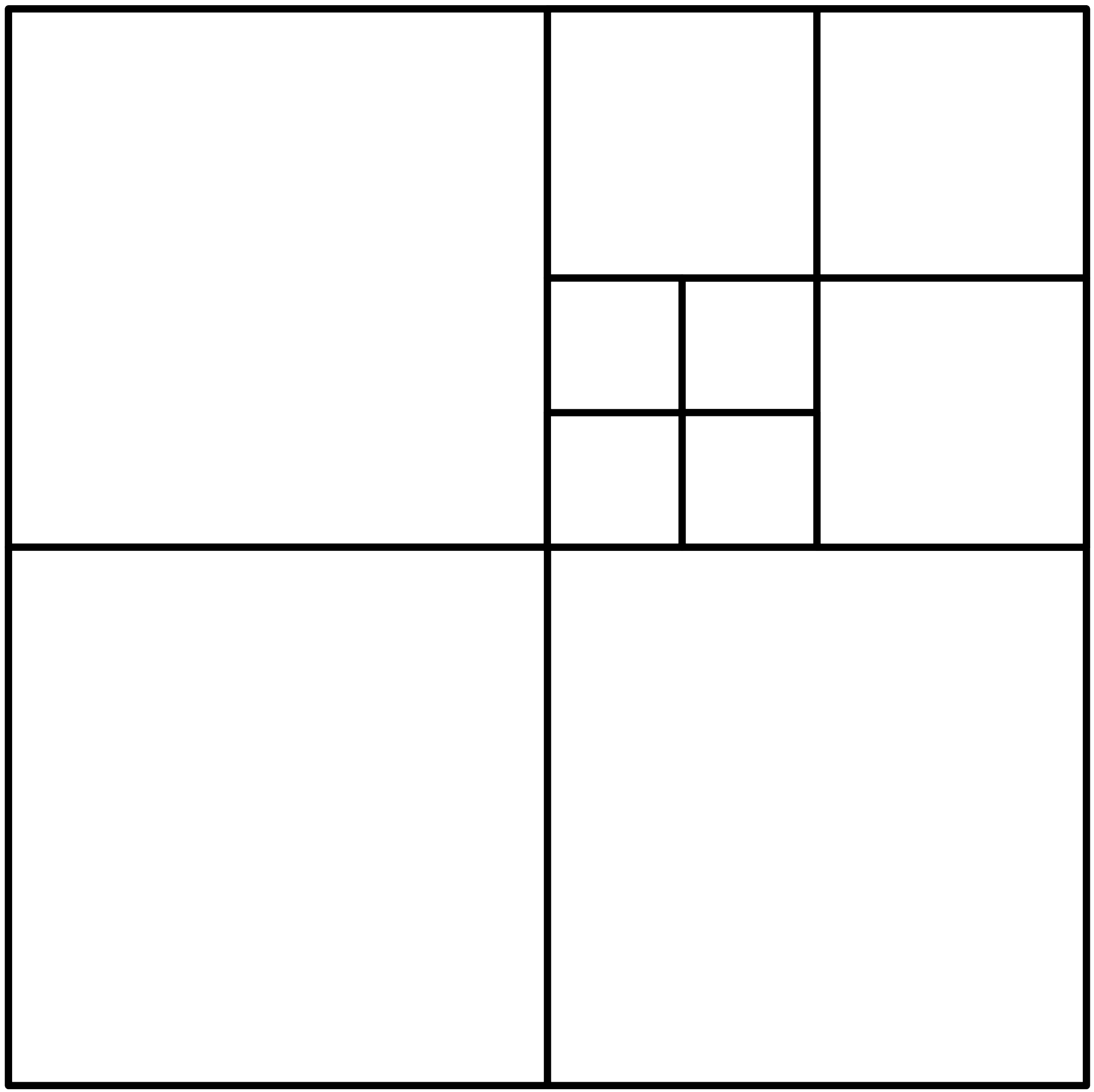

An Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

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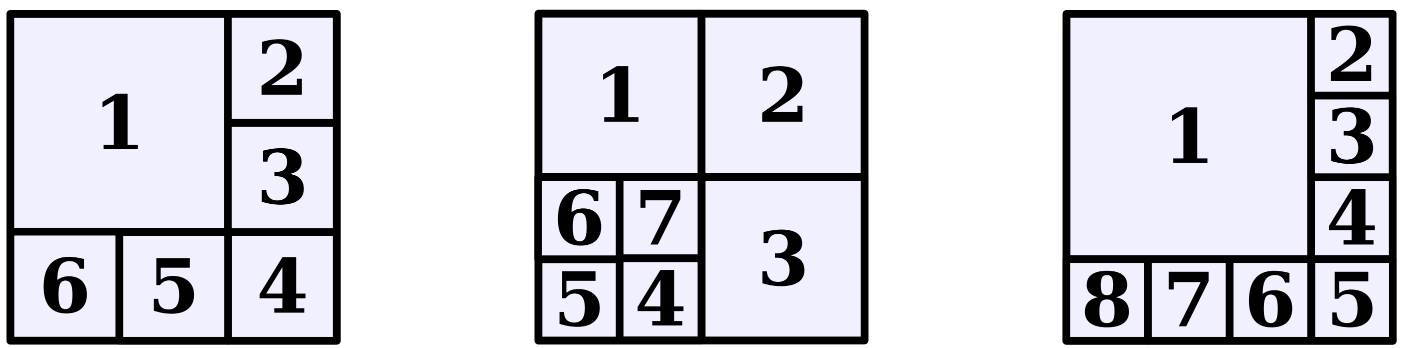

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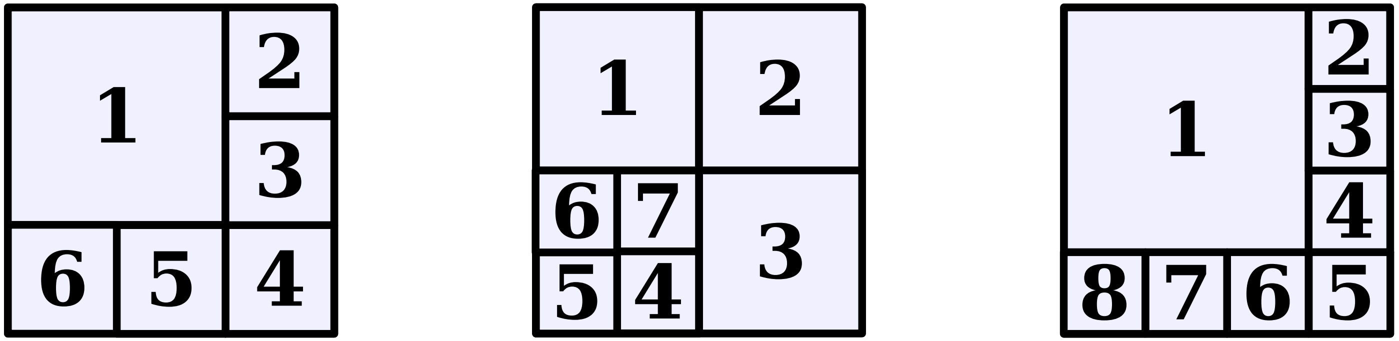

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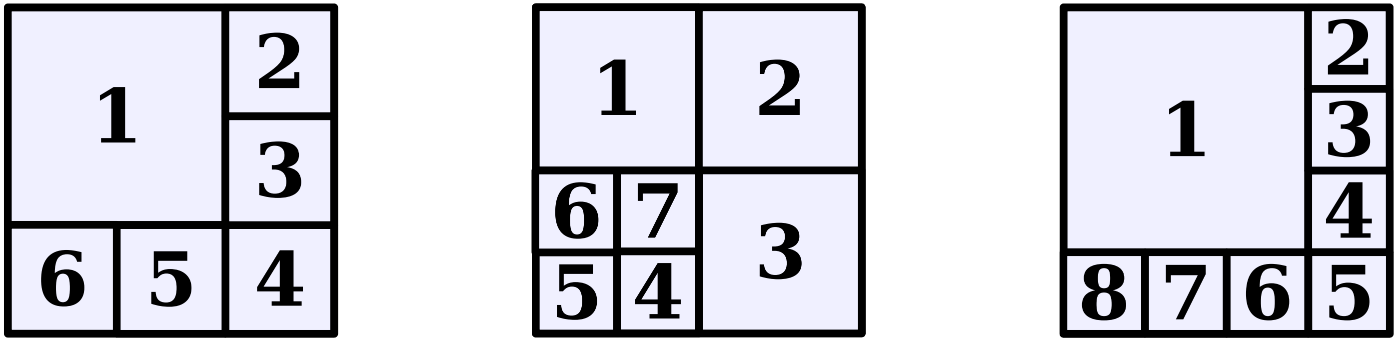

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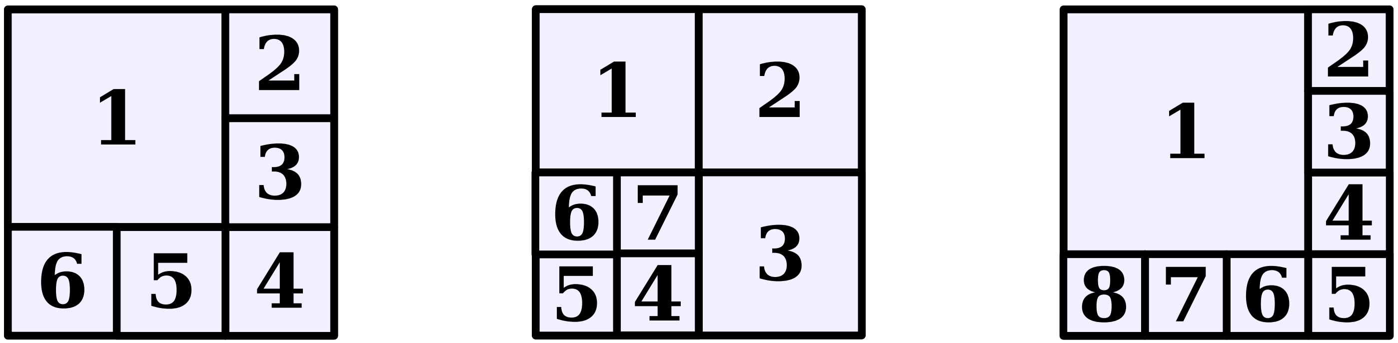

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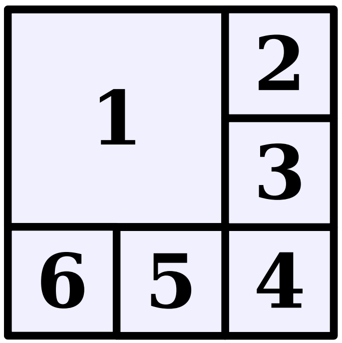

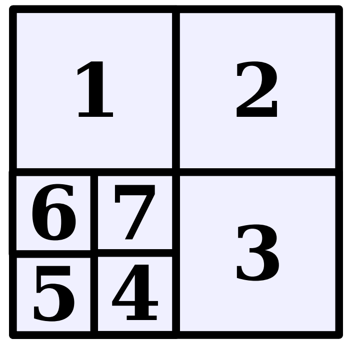

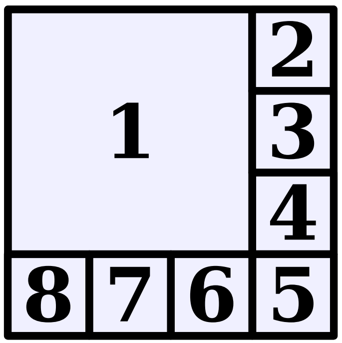

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Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

Proof: Let $P(n)$ be the statement "there is a way to subdivide a square into n smaller squares." We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

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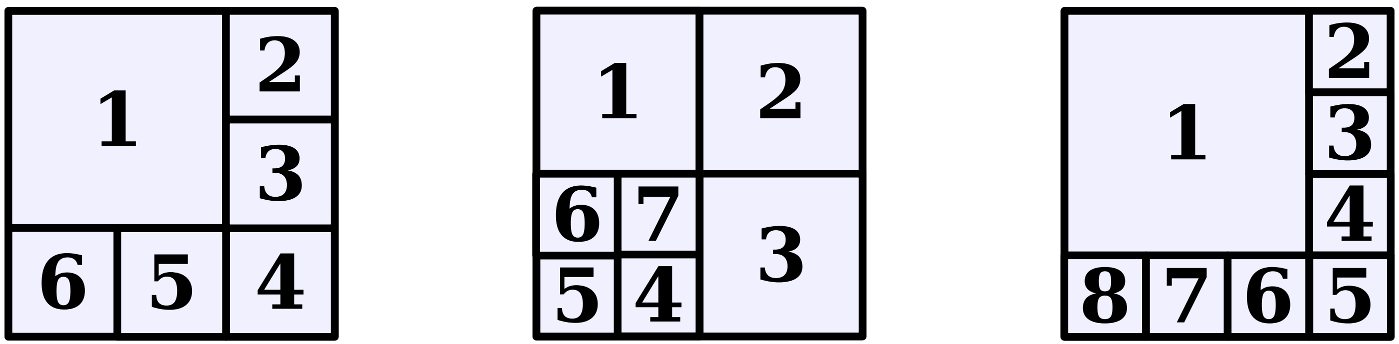

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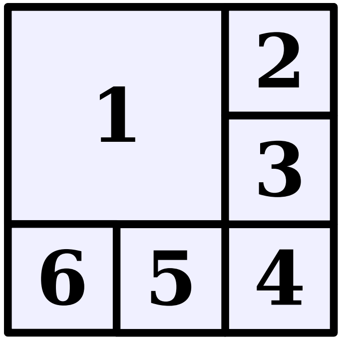

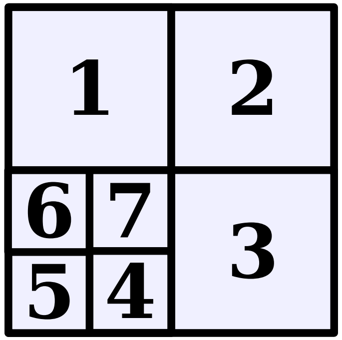

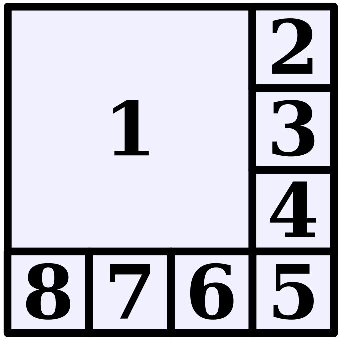

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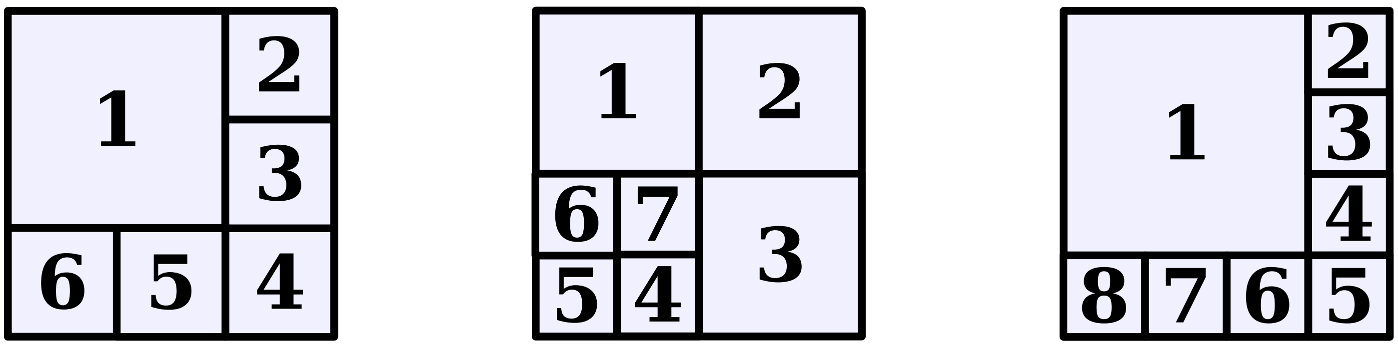

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Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that…
  - … you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - … you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you.

More on Square Subdivisions

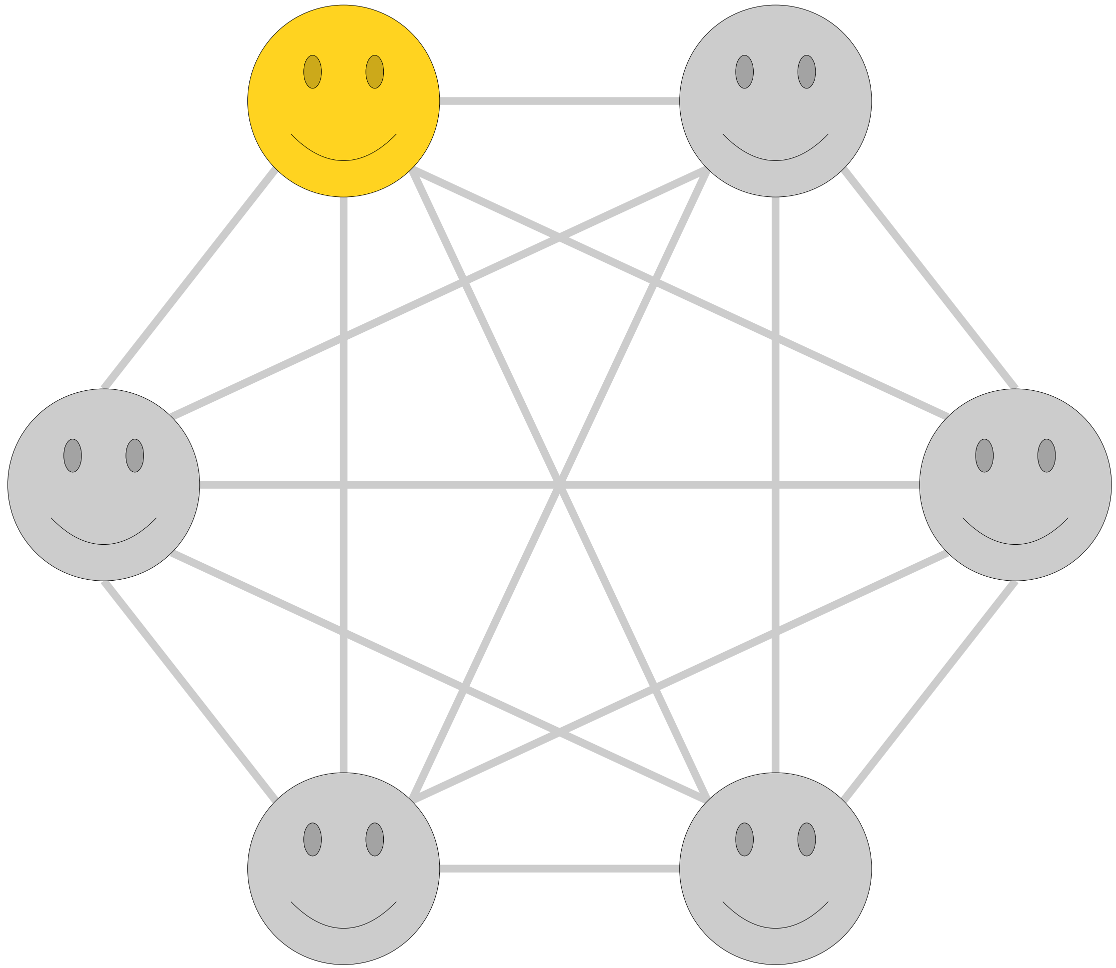
- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

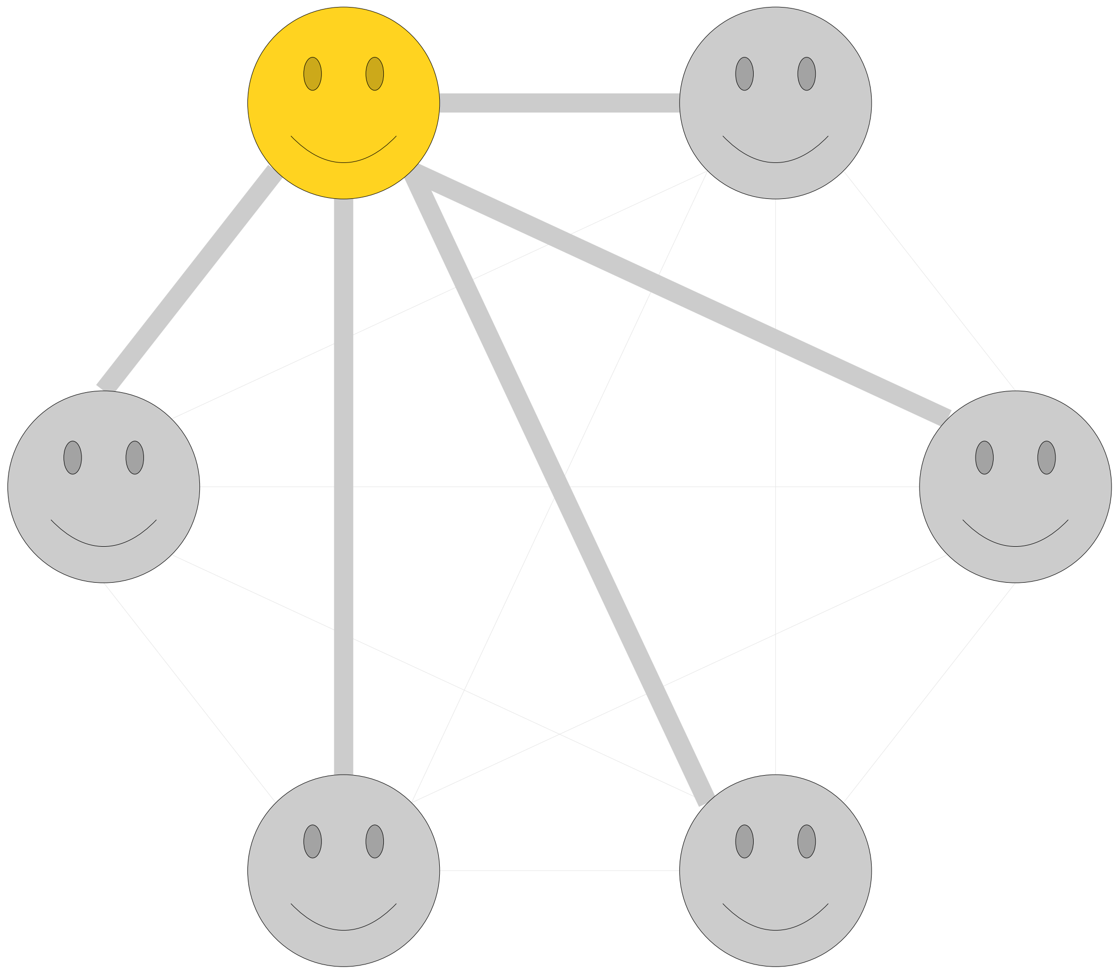
Ramsey Revisited

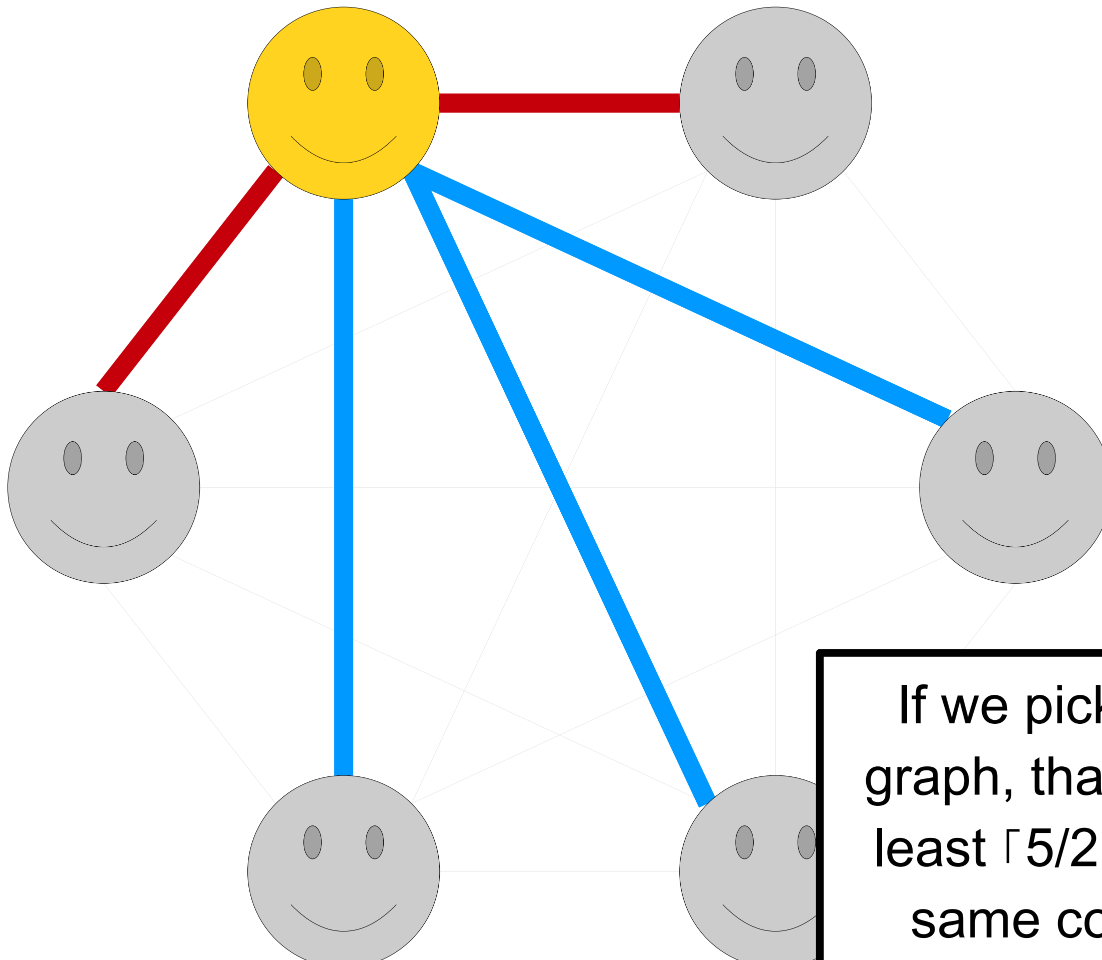
Ramsey Revisited

- In lecture, we proved the Theorem on Friends and Strangers: any 6-clique whose edges are painted one of two colors contains a monochrome triangle.
- On PS4, you're proving that any 17-clique whose edges are painted one of three colors has a monochrome triangle.
- What about if you use four colors? Five colors? Six colors?

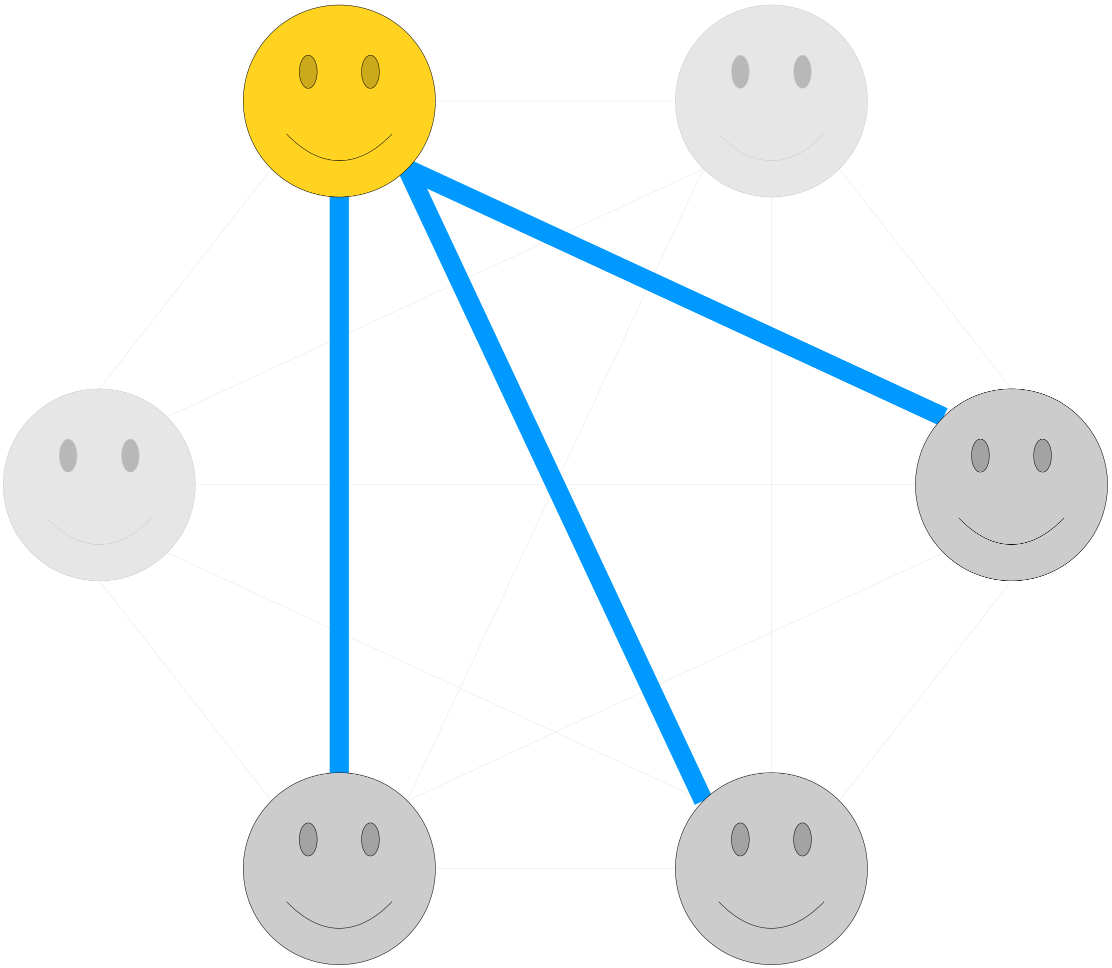
Refresher on Friends and Strangers

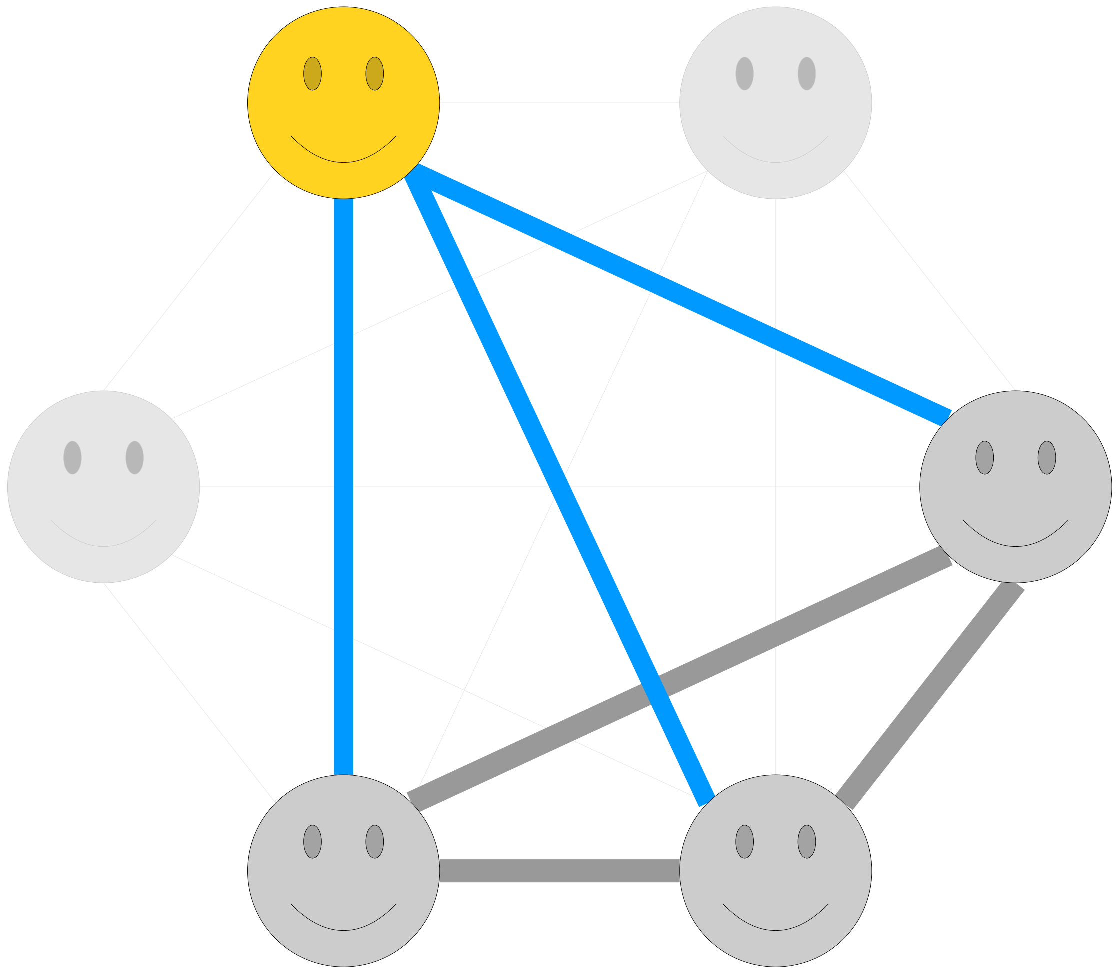


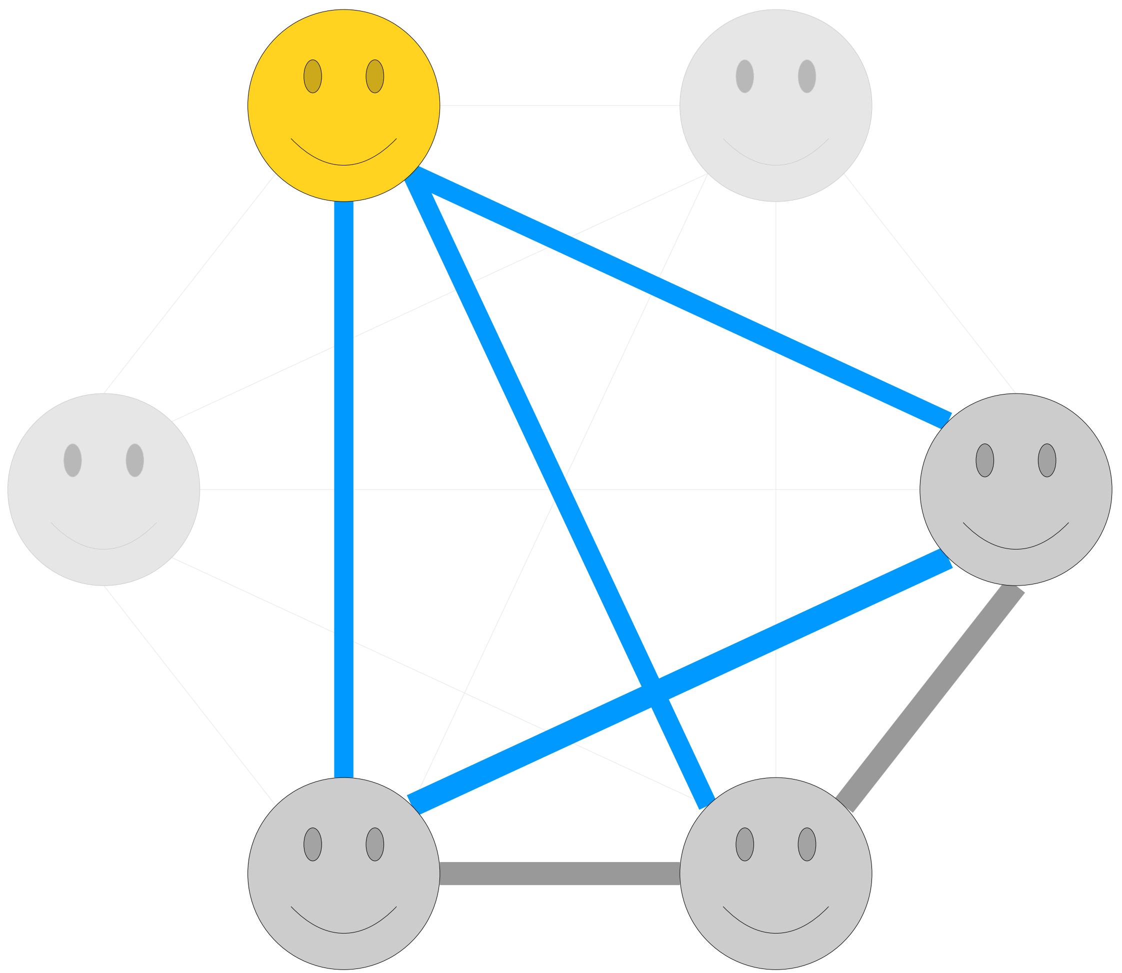


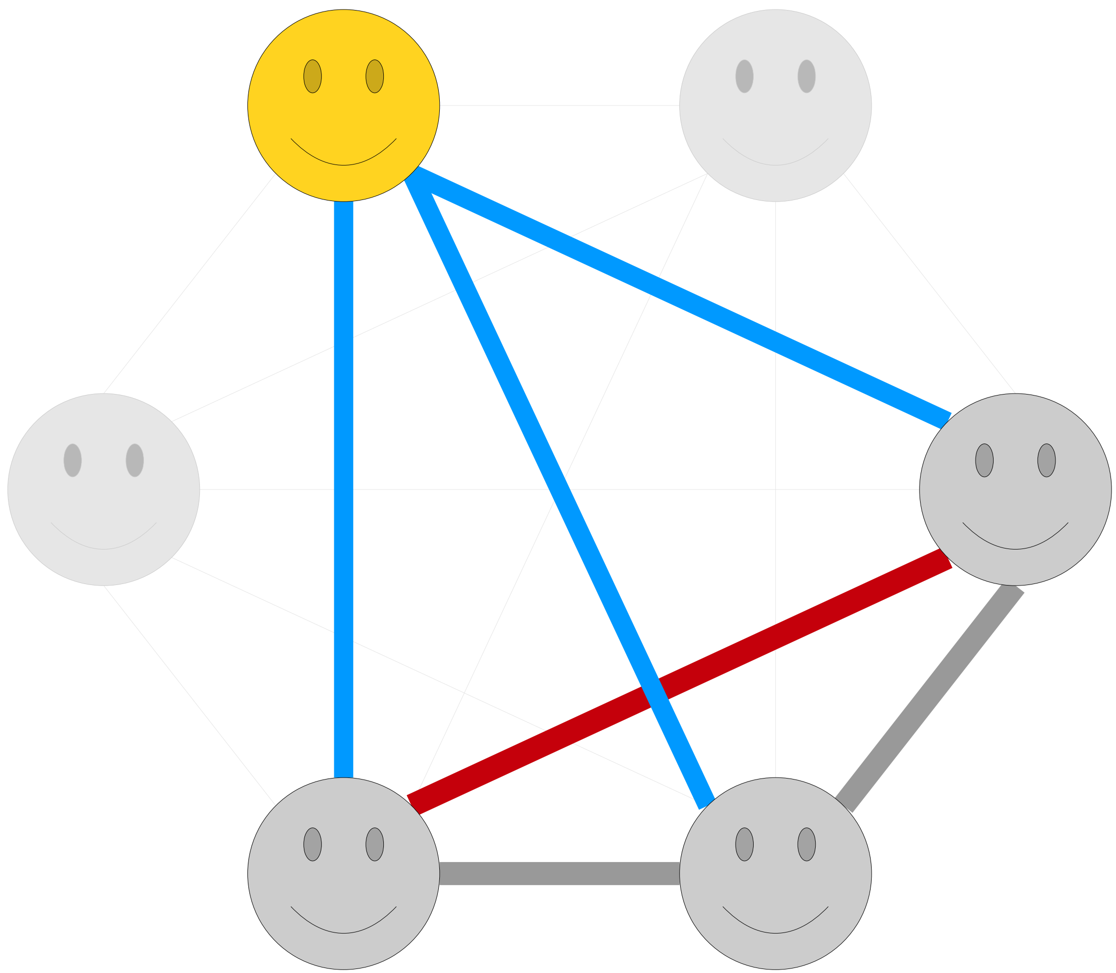


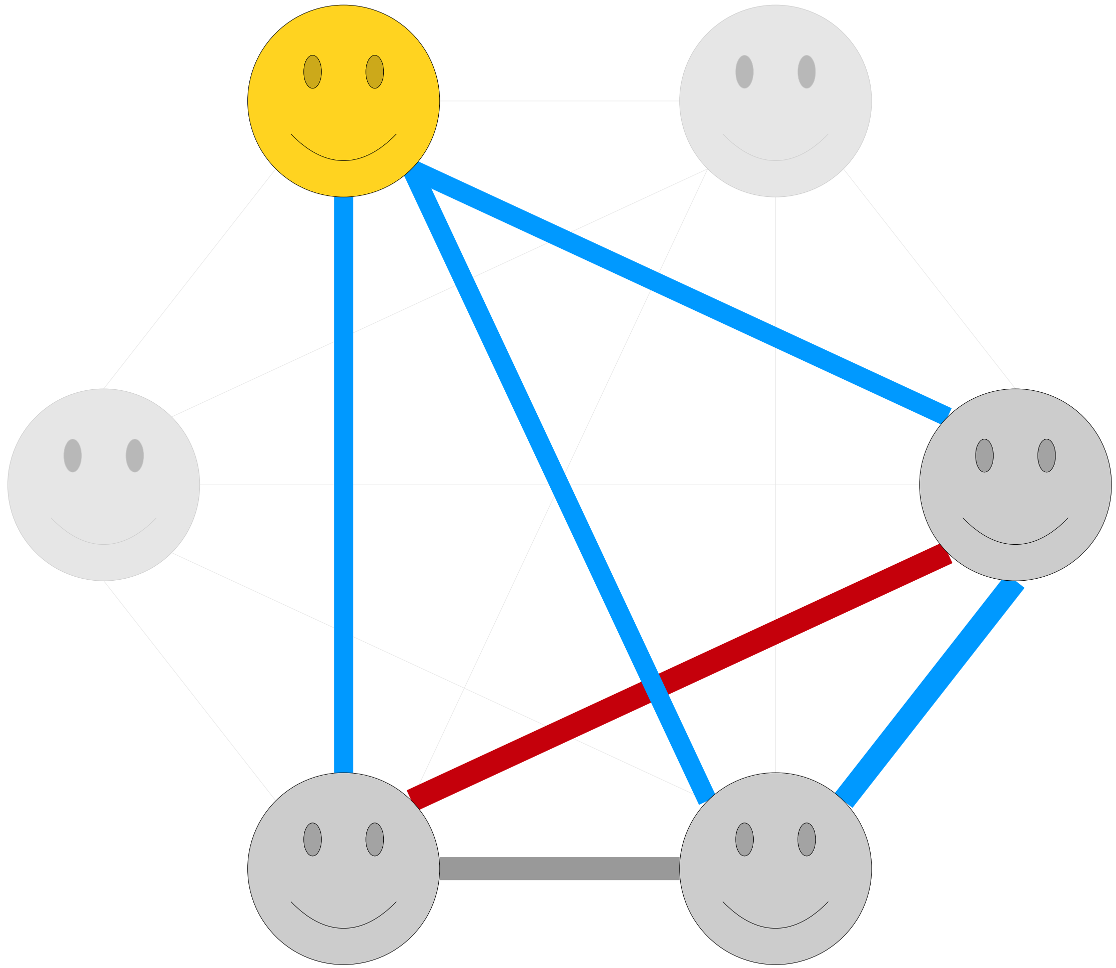
If we pick any node in the graph, that node will have at least $\lceil 5/2 \rceil = 3$ edges of the same color incident to it.

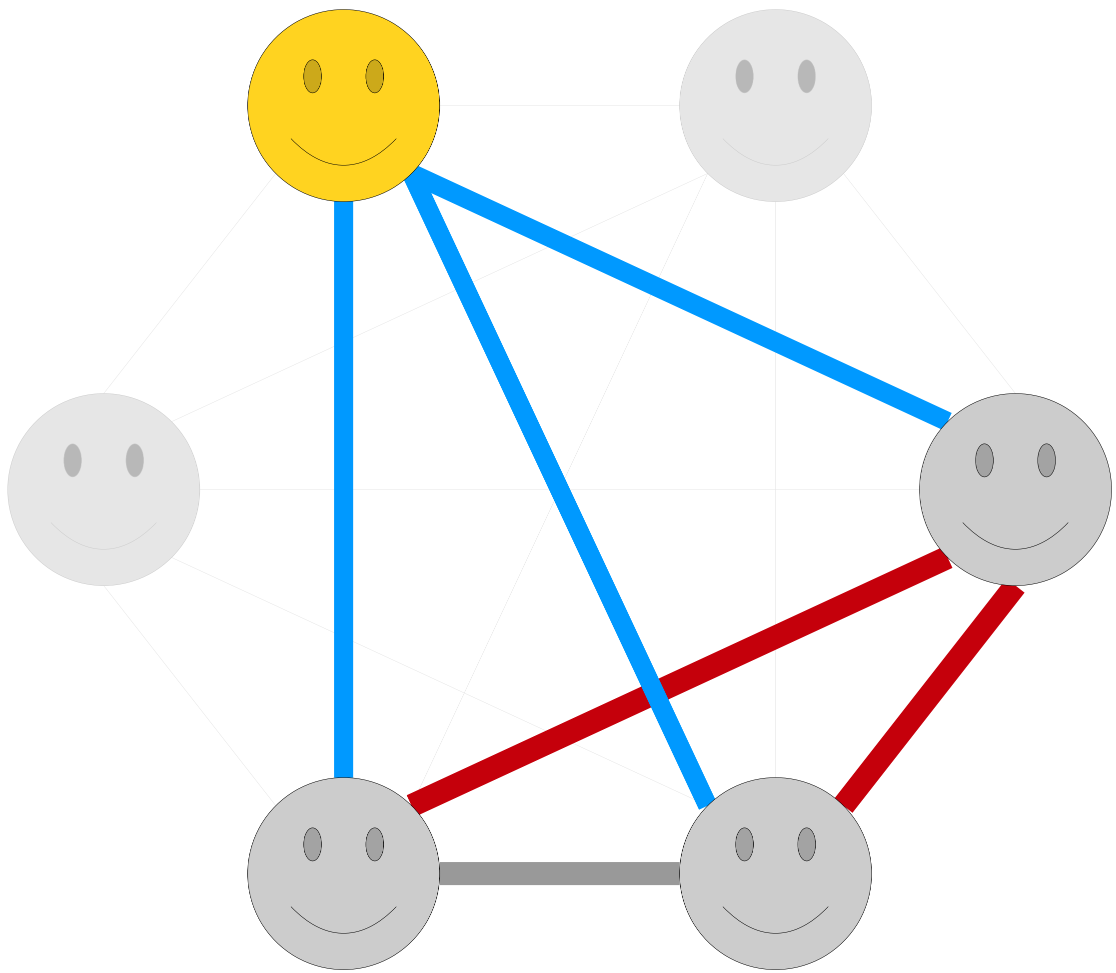


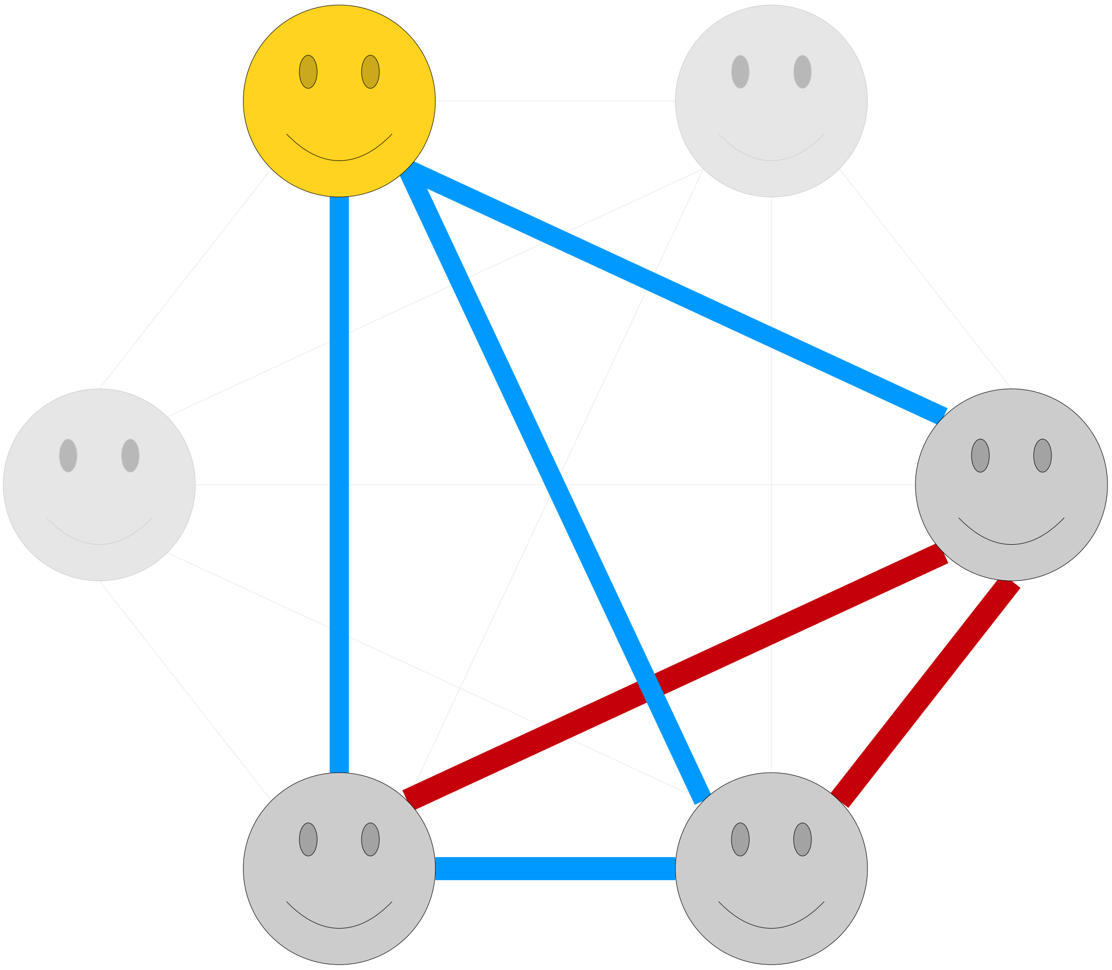


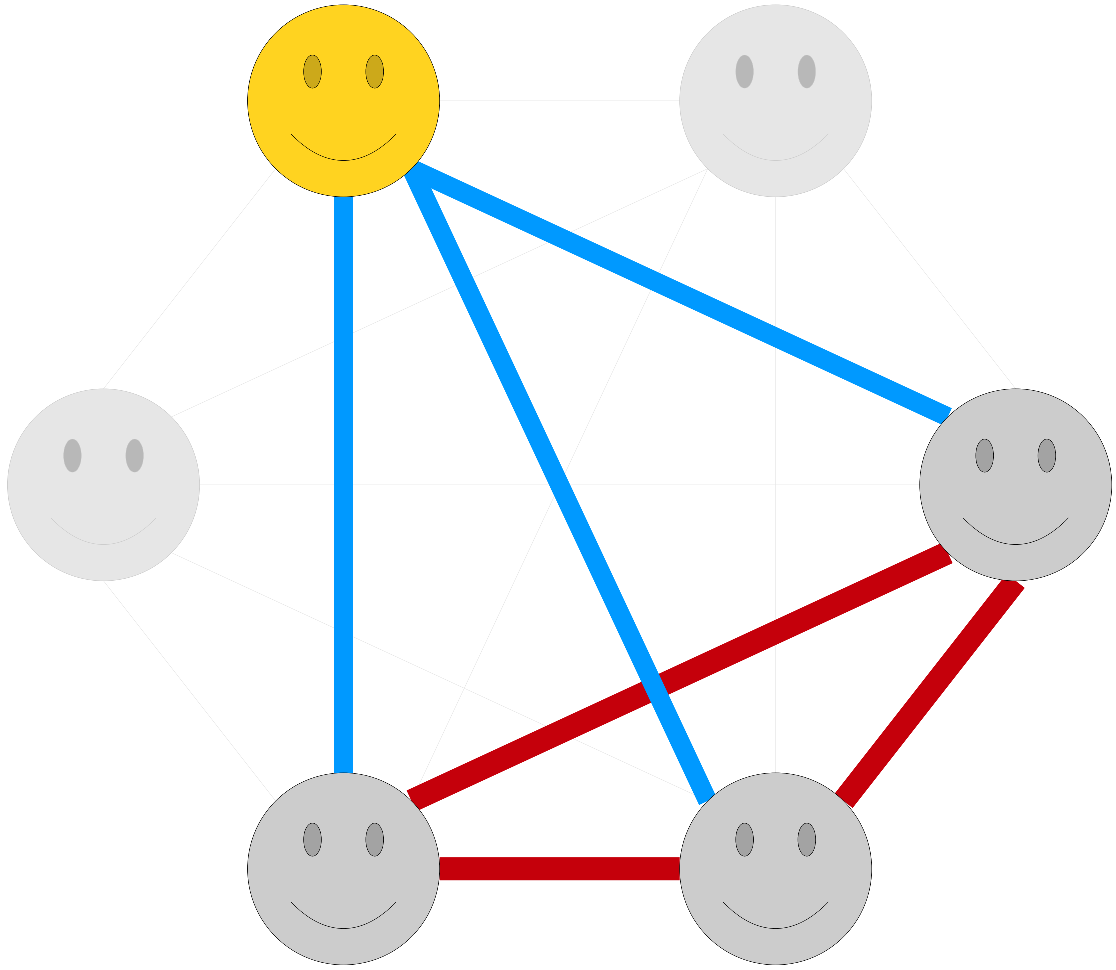












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The notation $n!$ represents ***n factorial***, the product of all natural numbers between 1 and n, inclusive.

$$5! = 1 \times 2 \times 3 \times 4 \times 5.$$

The value $3n!$ is read as $3(n!)$.

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As a base case, we prove $P(1)$.

Based on this choice of $P(n)$, what are we trying to prove in the base case?

Respond at pollev.com/zhenglian740

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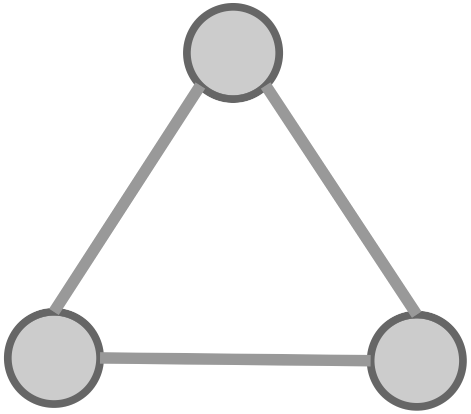

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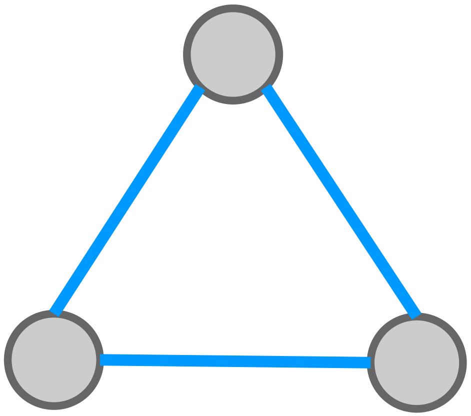

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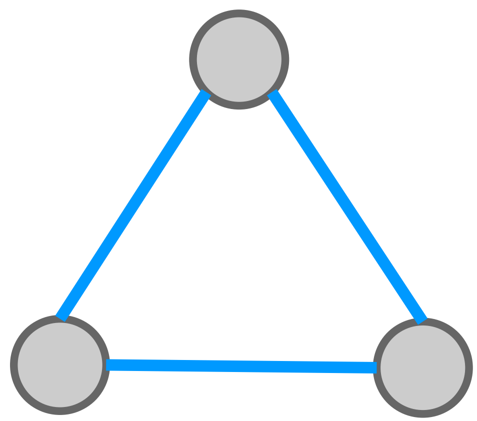

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Next, pick a natural number $k \geq 1$ and assume $P(k)$ is true

Based on this choice of $P(n)$, what are we assuming in the inductive step?

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To prove $P(k+1)$, what should we do?

A) Pick a $3k!$-clique with edges colored using k colors, apply the inductive hypothesis, then add in nodes to create a larger $3(k+1)!$-clique.
B) Pick a $3(k+1)!$-clique with edges colored using $k+1$ colors, then discover a smaller $3k!$-clique within that larger clique to apply the inductive hypothesis to.
C) Both options work.
D) Neither option works.

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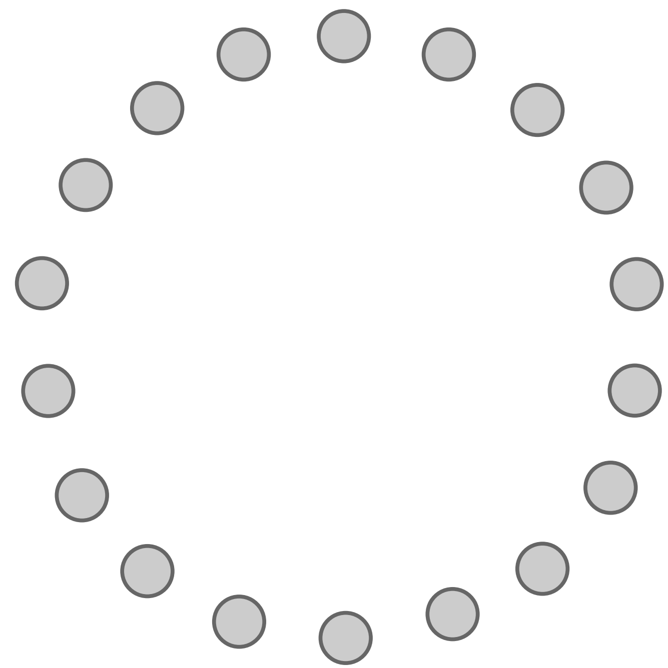

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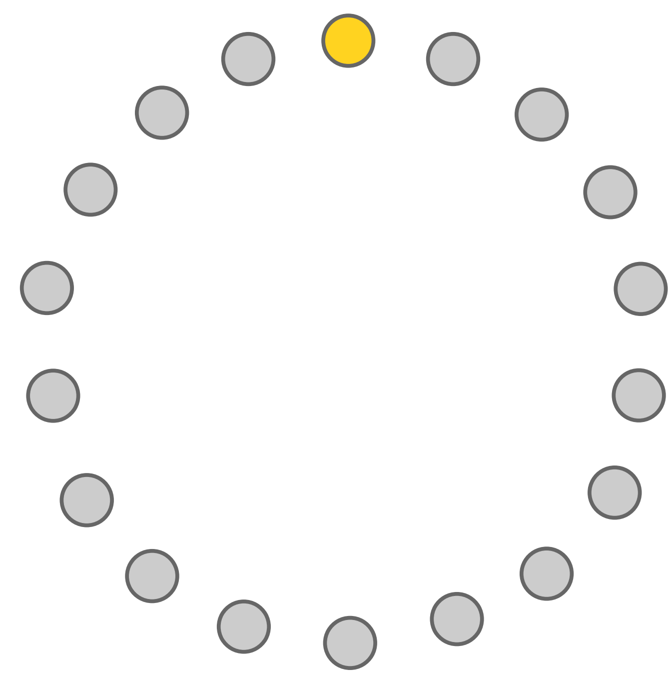
How many edges have an endpoint at this node?

How many possible edge colors do we have?



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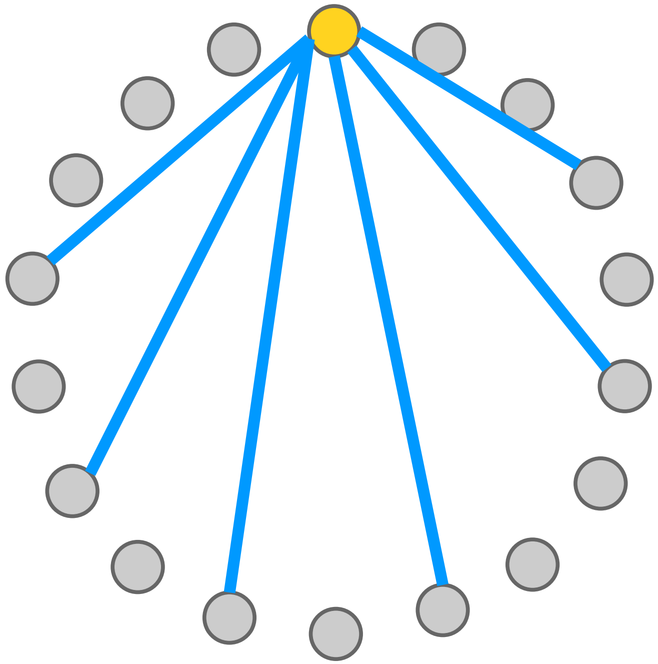

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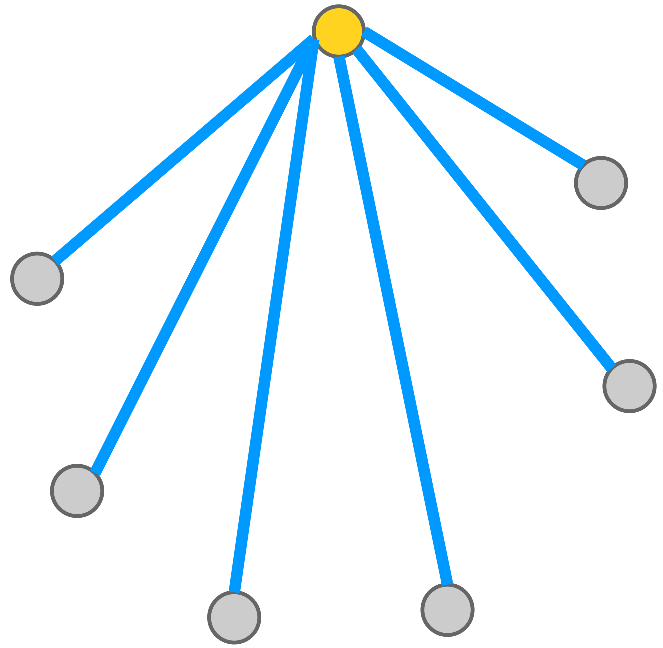
Now let's look at these nodes that are adjacent to our chosen node via a blue edge. How do they relate to one another?



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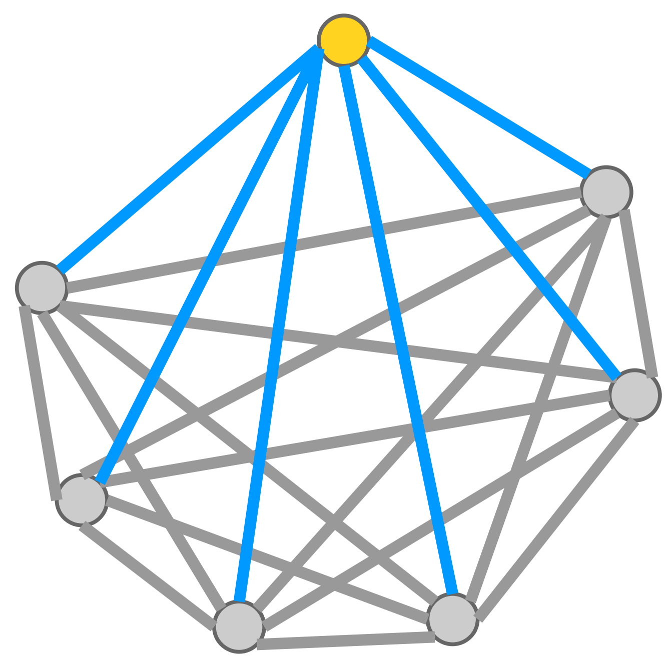
Note: I'm coloring these edges in grey to indicate that we don't know how these edges are colored, just that it's some arbitrary coloring from the k+1 possible colors.



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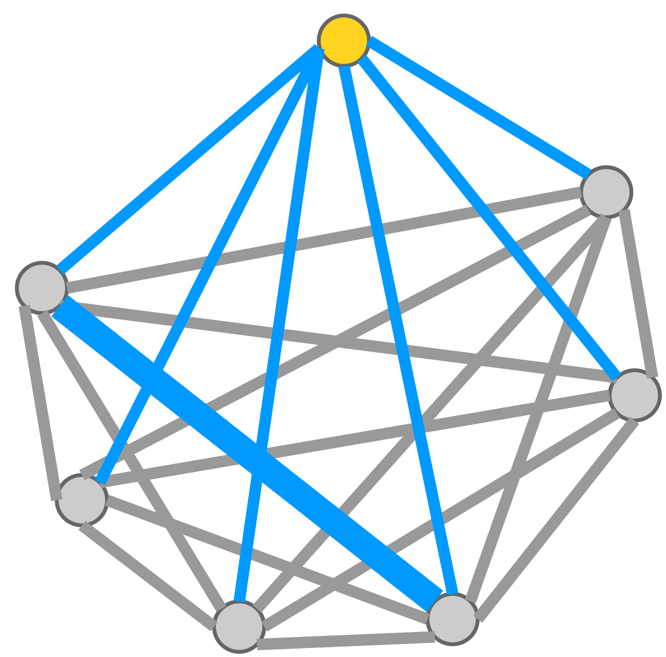
Observation: if any one of these edges is blue, then we've found a blue triangle and we're done.



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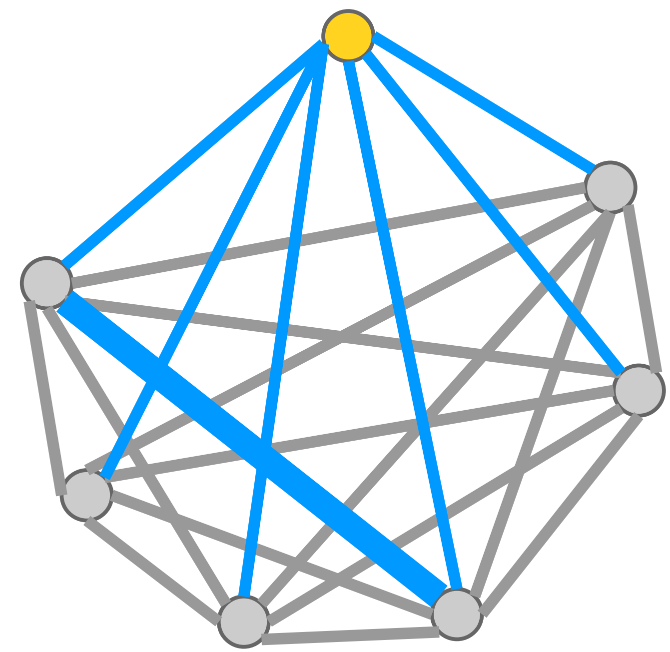
So let's suppose that <u>*none*</u> of these edges are blue. What happens in that case?



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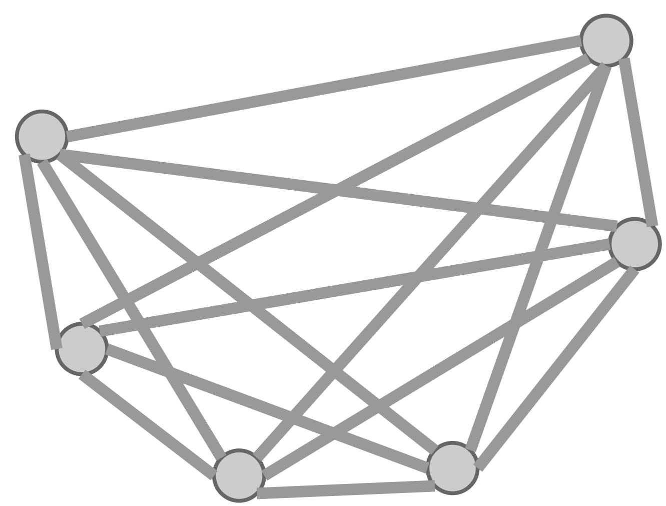

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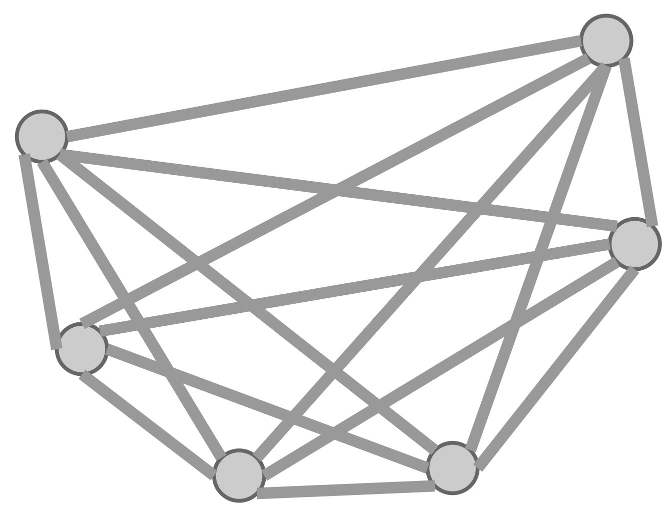
Hey look, it's a clique! How many nodes does it have? How many possible colors are there for the edges? What does our inductive hypothesis say about cliques of that size?



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Pick any node v in the clique and look at the edges incident to v. There are $3(k+1)! - 1$ other nodes in the clique and $k+1$ colors. By the generalized pigeonhole principle, this means v is adjacent to at least

$$\left\lceil \frac{3(k+1)! - 1}{k+1} \right\rceil$$

nodes by edges of the same color.

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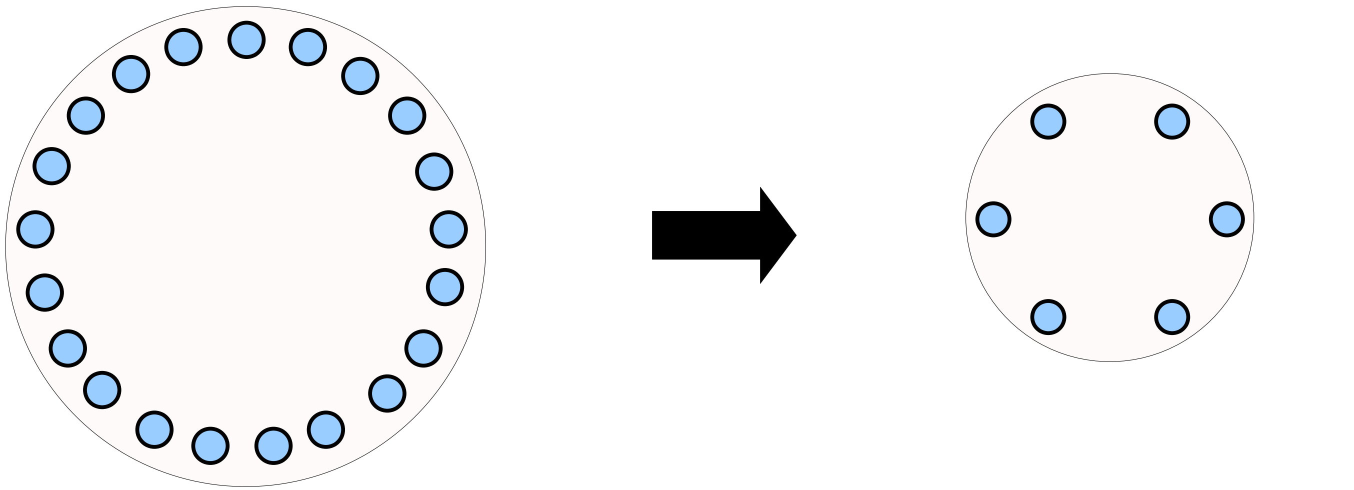
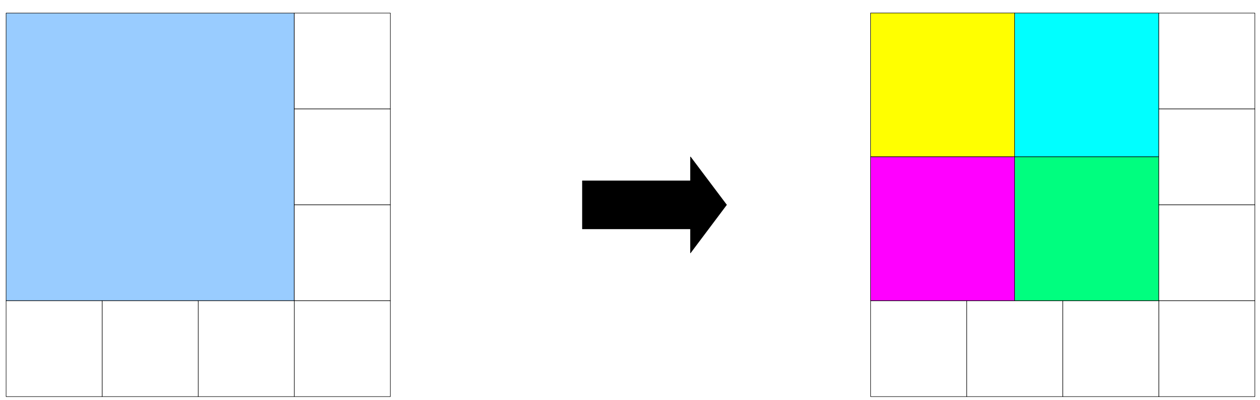
An Observation



Start with larger clique

Get to smaller clique



Start with fewer squares

Get to more squares

Following the Rules

- When working with square subdivisions, our predicate looked like this:

$P(n)$ is "***there exists*** a way to subdivide a square into n squares."

- When working with cliques, our predicate looked like this:

$P(n)$ is "***for any*** coloring of a $3n!$-clique, there is a monochrome triangle."

- With squares, the quantifier is ∃. With cliques, the first quantifier is ∀.
- This fundamentally changes the "feel" of induction.

Build Up with ∃

- In the case of squares, in our inductive step, we prove

  If

  there exists a subdivision into k squares,

  then

  there exists a subdivision into k+3 squares.

- Assuming the antecedent gives us a concrete subdivision into k squares.
- Proving the consequent means finding some way to subdivide in to k+3 squares.
- The inductive step goal is to "***build up***:" start with a smaller number of squares, and somehow work out what to do to get a larger number of squares.

Build Down with ∀

- In the case of cliques, in our inductive step, we prove

  If

  for all colorings of a $3k!$-clique, there's a mono. tri.

  then

  for all colorings of a $3(k+1)!$-clique, there's a mono. tri.

- Assuming the antecedent means once we find a k-colored $3k!$-clique, we get a monochrome triangle.
- Proving the consequent means picking an arbitrary coloring of a $3(k+1)!$-clique, then trying to find a triangle in it.
- The inductive step goal is to "***build down***:" start with a larger clique, then find a way to turn it into a smaller clique.

More on Ramsey Triangles

- We've proved that $3n!$ nodes is enough to get a triangle with $n \geq 1$ colors on the edges.
- For $n = 3$, this says we need 18 nodes, on PS4 you'll prove that you can do this with just 17 nodes.
- For $n = 4$, this says we need 72 nodes. We know that 50 nodes is too few and 66 nodes is enough. The actual answer is therefore somewhere between 51 and 66.
- ***Open problem:*** Find the exact minimum number of nodes needed to get a monochrome triangle with $n \geq 4$ colors.
- ***Challenge problem:*** Show that $\lceil e \cdot n! \rceil$ nodes is always sufficient to get a monochrome triangle with $n \geq 1$ colors. *(This is hard but doable if you know the material from CS103, plus the Taylor series for e.)*

Let's take a quick break!

Time-Out for Announcements!

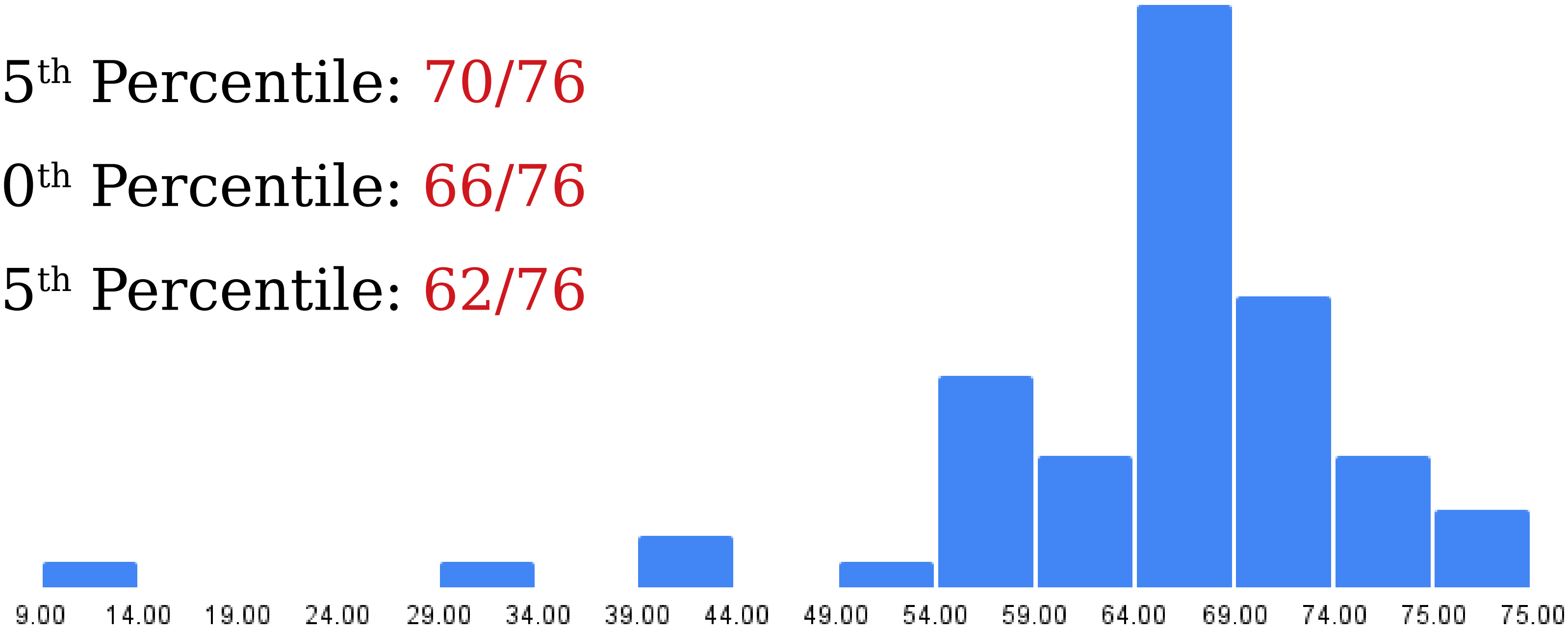
Problem Set Two Graded

- Your diligent and hardworking TAs have finished grading PS2. Grades and feedback are now available on Gradescope.

75th Percentile: 70/76

50th Percentile: 66/76

25th Percentile: 62/76



- As always, ***please review your feedback!*** Knowing where to improve is more important than just seeing a raw score.
- Did we make a mistake? Regrades are open and are due by next Thursday.

Problem Sets

- Problem Set Three was due today at 5:30PM.
- Problem Set Four goes out today. It's due next Friday at 5:30PM.
  - Because this coincides with the day of the midterm, we are implementing the following policy:
    - On-time submissions will receive a small bonus (5%).
    - There is a penalty-free 48 hour grace period to submit until Sunday at 5:30PM.
    - This policy applies for this assignment only.

Midterm Exam Logistics

- Our midterm exam will be on Friday, July 26th from 5:00 – 8:00 PM in Hewlett 201 (our normal lecture room).
- You're responsible for lectures up to the end of week 3 and topics from PS1 – PS3. Later lectures and problem sets won't be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is open-book, open-note, and closed-other-humans/AI.

Midterm Accommodations

- This is your ***last call*** for midterm accommodations:
    - If you have OAE accommodations, you should have received an email from us with exam time and location.
    - If you have a midterm conflict, you should have received an email from us with instructions on how you will be taking the exam.
    - If you fall into either of these categories but have not heard from us, email the course staff *ASAP* at cs103-sum2324-staff@lists.stanford.edu.

Preparing for the Exam

- Review your assignment feedback and the solutions and make sure you understand our comments.
- Practice Midterm 1 – slightly easier than our exam.
- Practice Midterm 2 – approximately the same difficulty as our exam.
- 30 Extra Practice Problems across all topics.
- Please do ***not*** read the solutions to a problem until you have worked through it.

Let’s get back to CS103!

Complete Induction

Let P be some predicate. The ***principle of complete induction*** states that if

If it starts true…

$P(0)$ is true

and

…and it stays true…

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

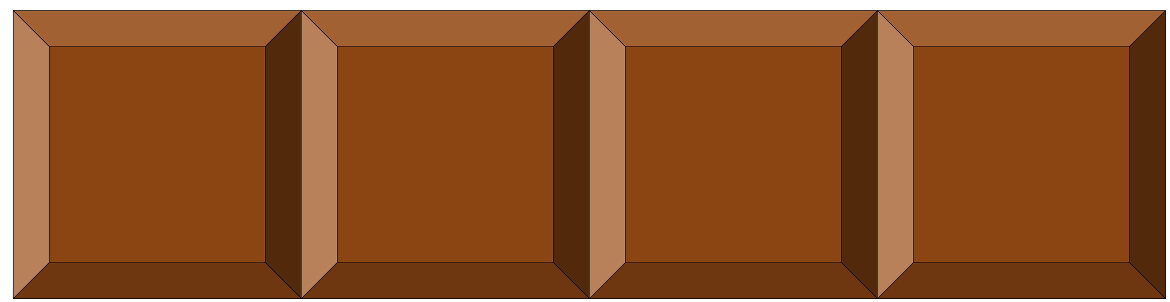
Mathematical Induction

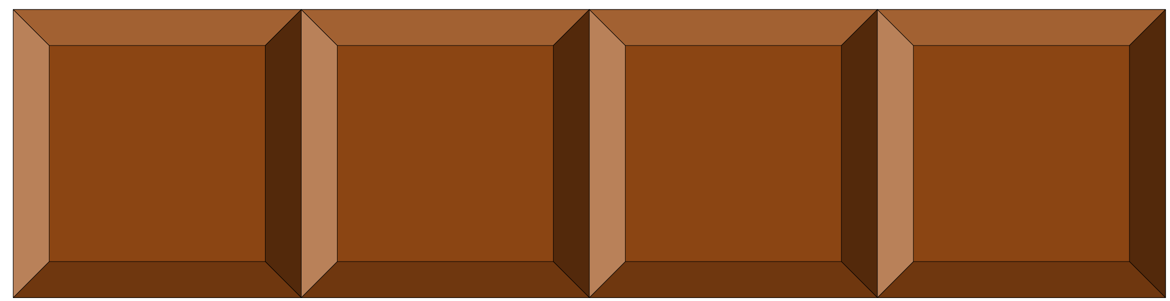
- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

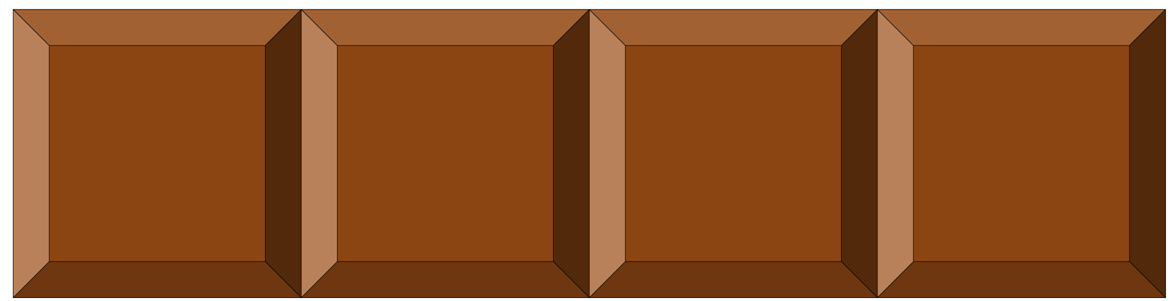
Complete Induction

- You can write proofs using the principle of ***complete*** induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that ***P*(0), *P*(1), *P*(2), …, and *P*(*k*)** are all true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

An Example: ***Eating a Chocolate Bar***

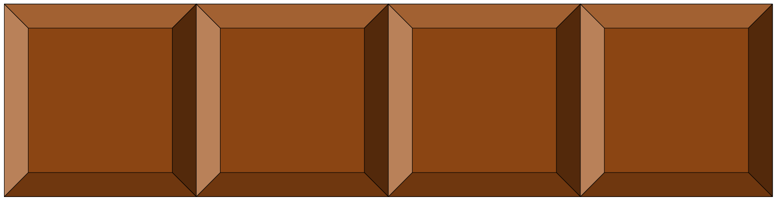


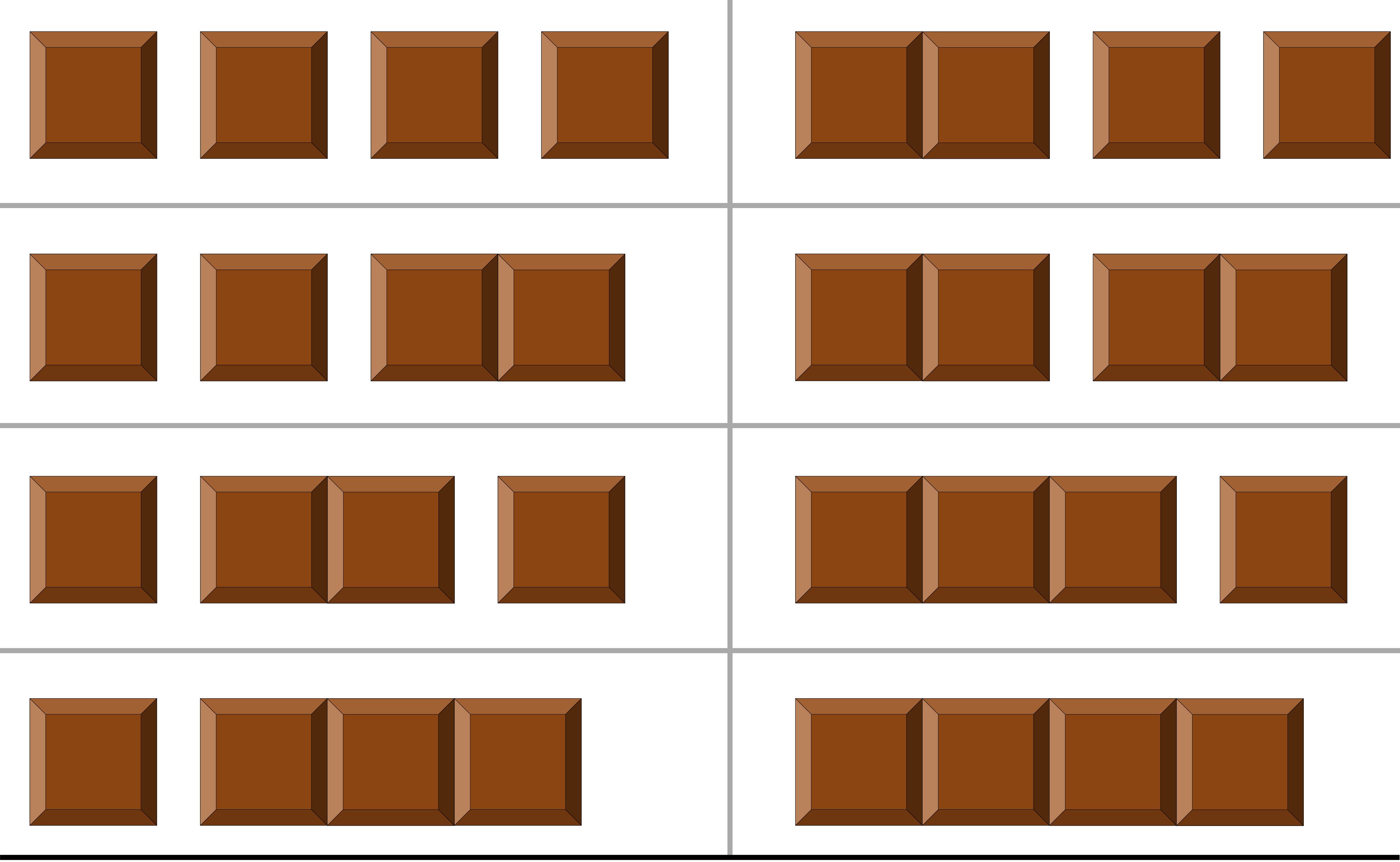




Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a…
  - 1×1 chocolate bar?
  - 1×2 chocolate bar?
  - 1×3 chocolate bar?
  - 1×4 chocolate bar?





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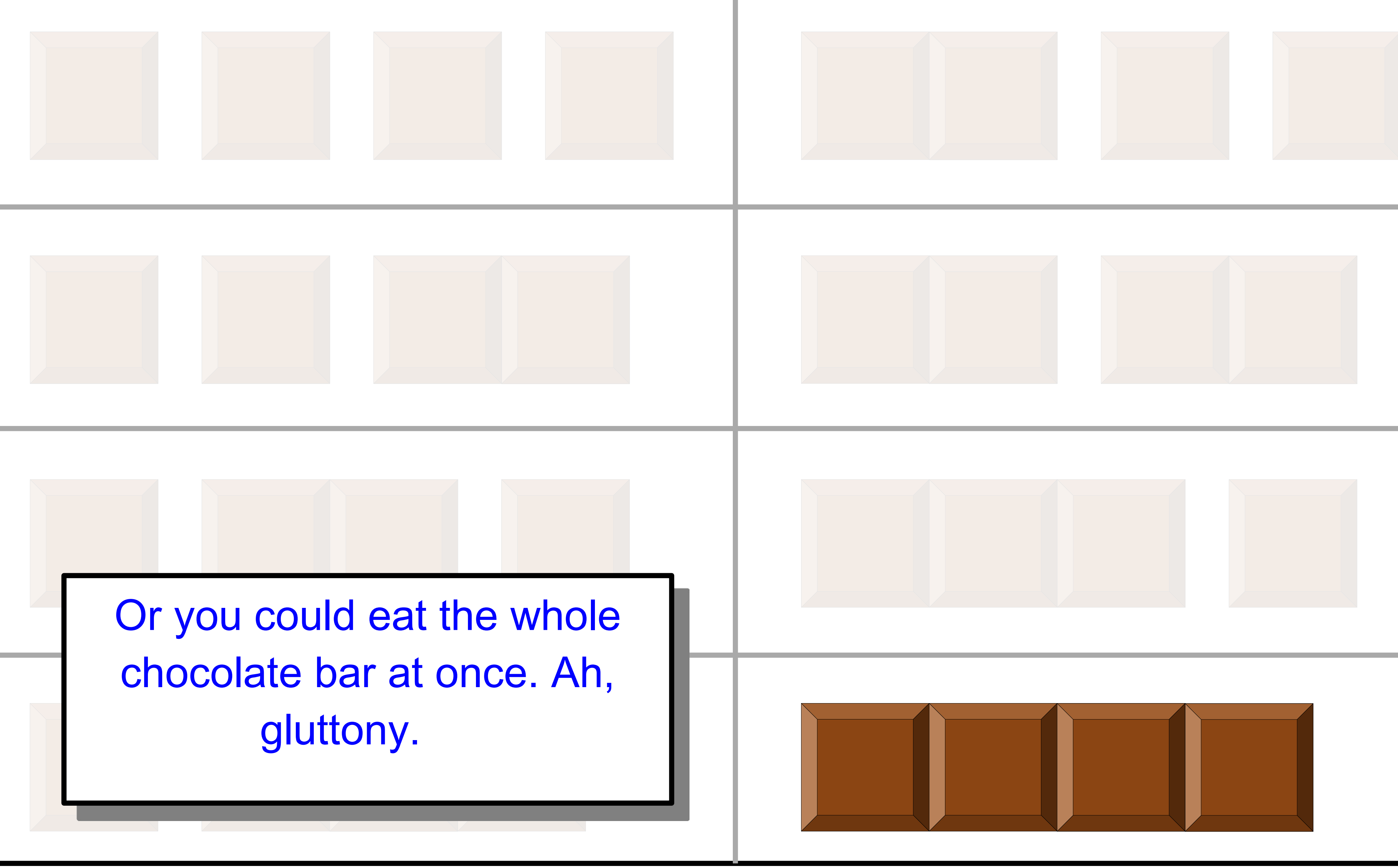

There are eight ways to eat a 1 × 4 chocolate bar.

Eating a Chocolate Bar

- There's…
  - 1 way to eat a 1×1 chocolate bar,
  - 2 ways to eat a 1×2 chocolate bar,
  - 4 ways to eat a 1×3 chocolate bar, and
  - 8 ways to eat a 1×4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size k, then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

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Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \ldots + 2^2 + 2^1 + 2^0$$

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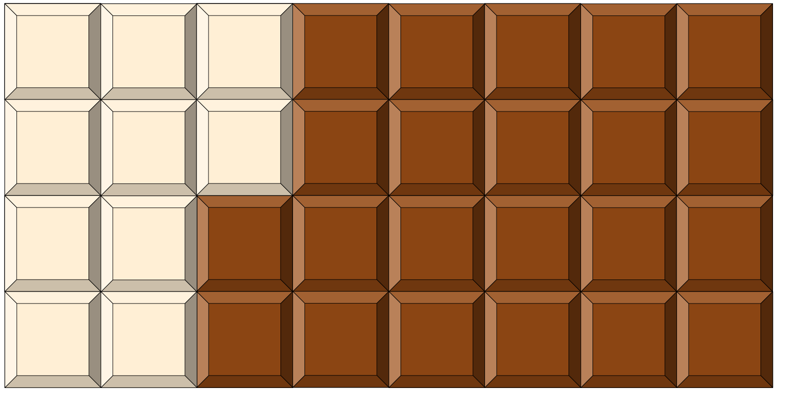
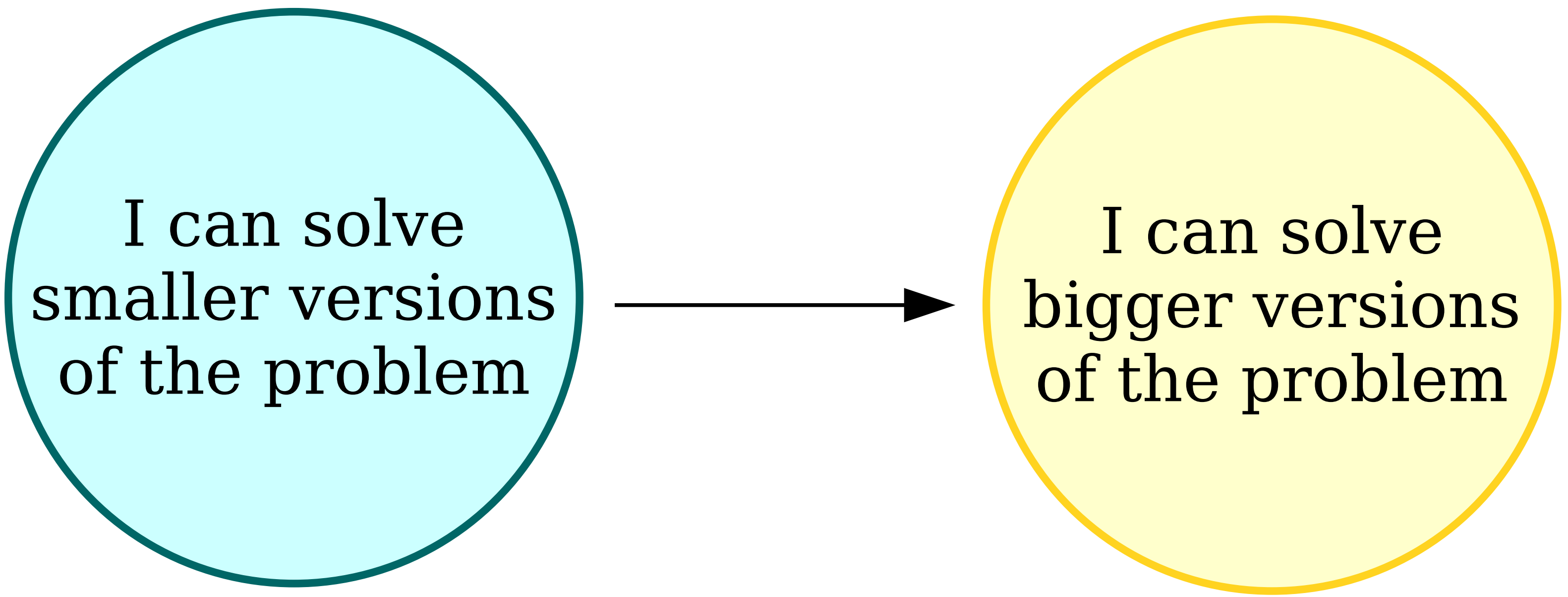
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?



- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

Induction vs. Complete Induction



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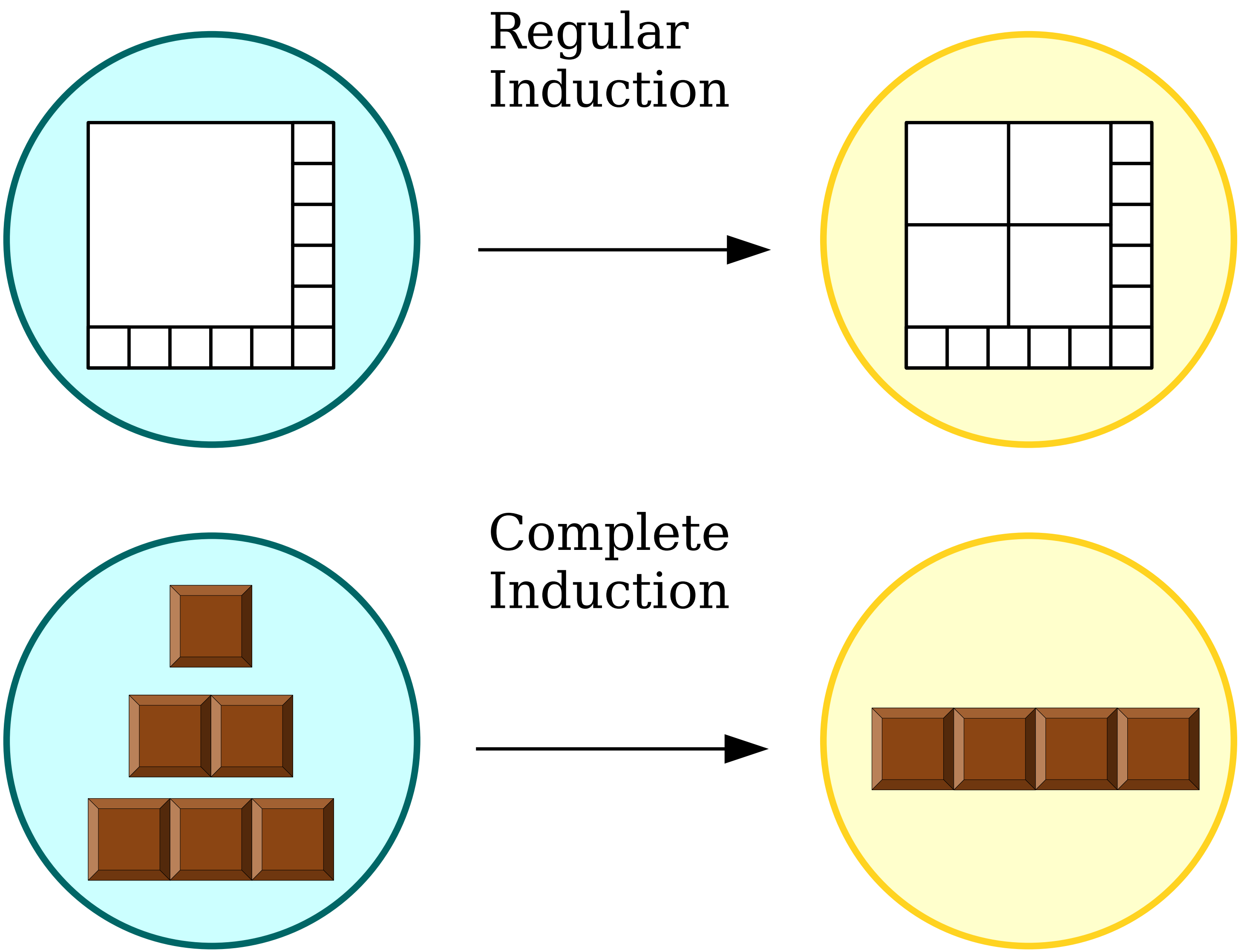

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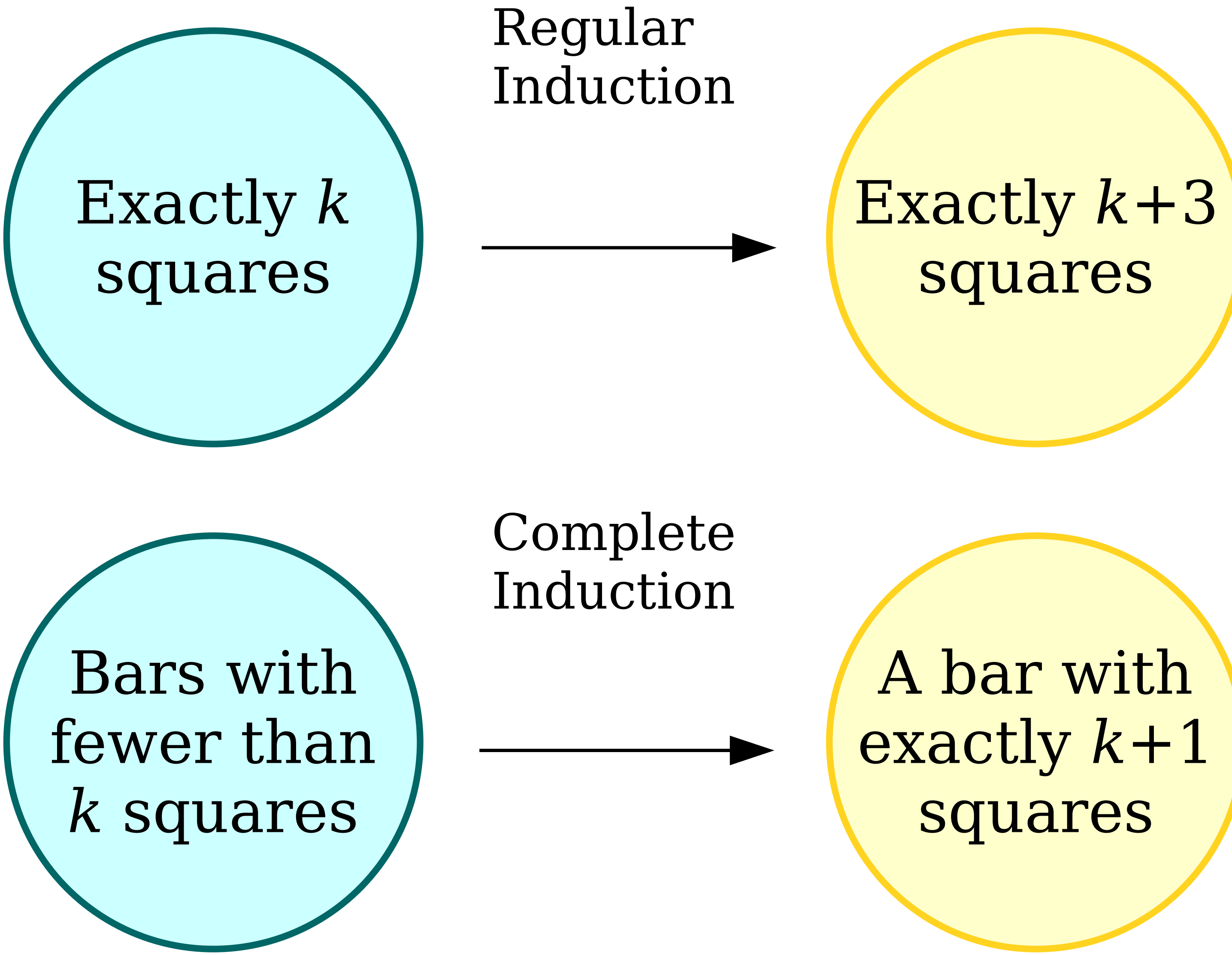

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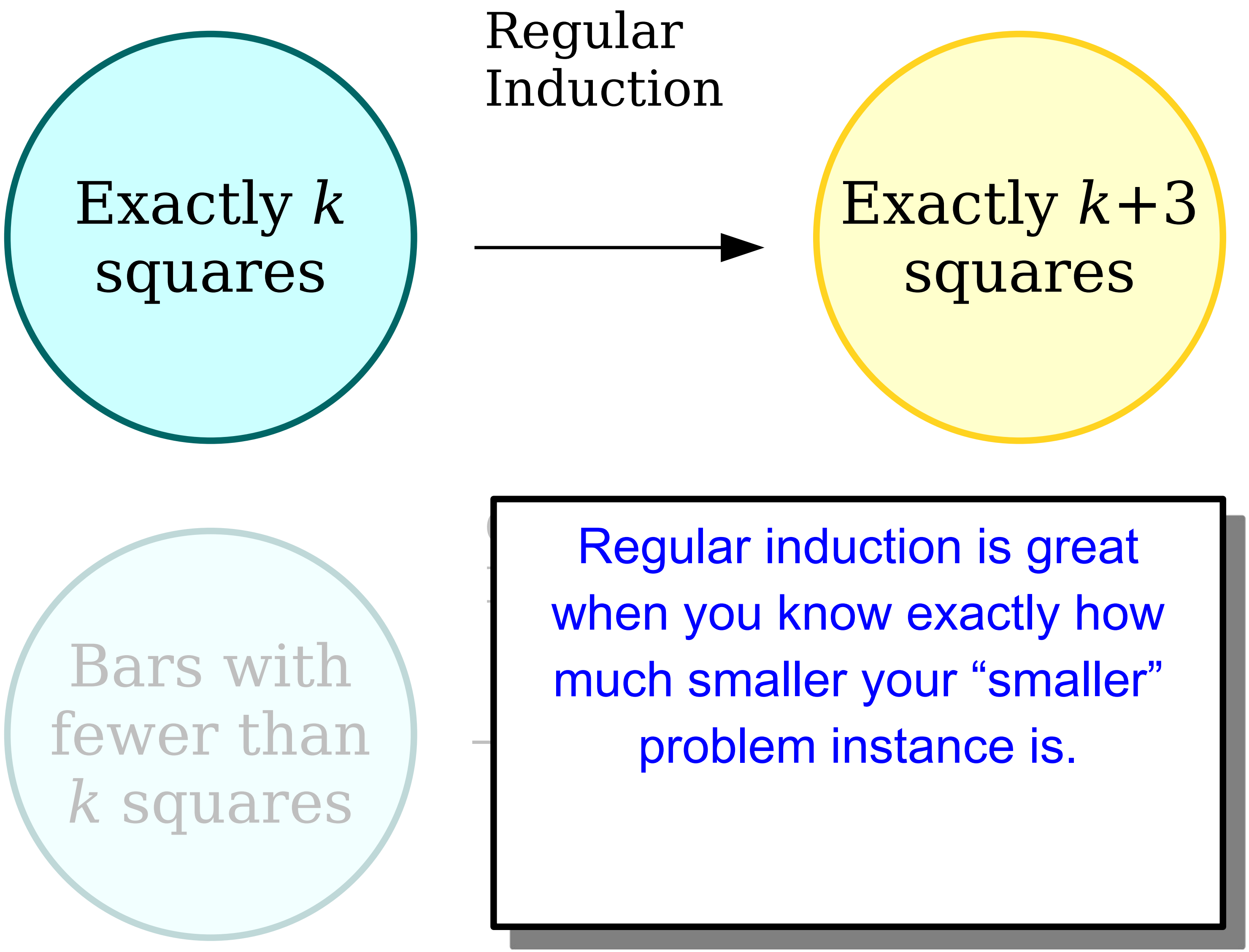

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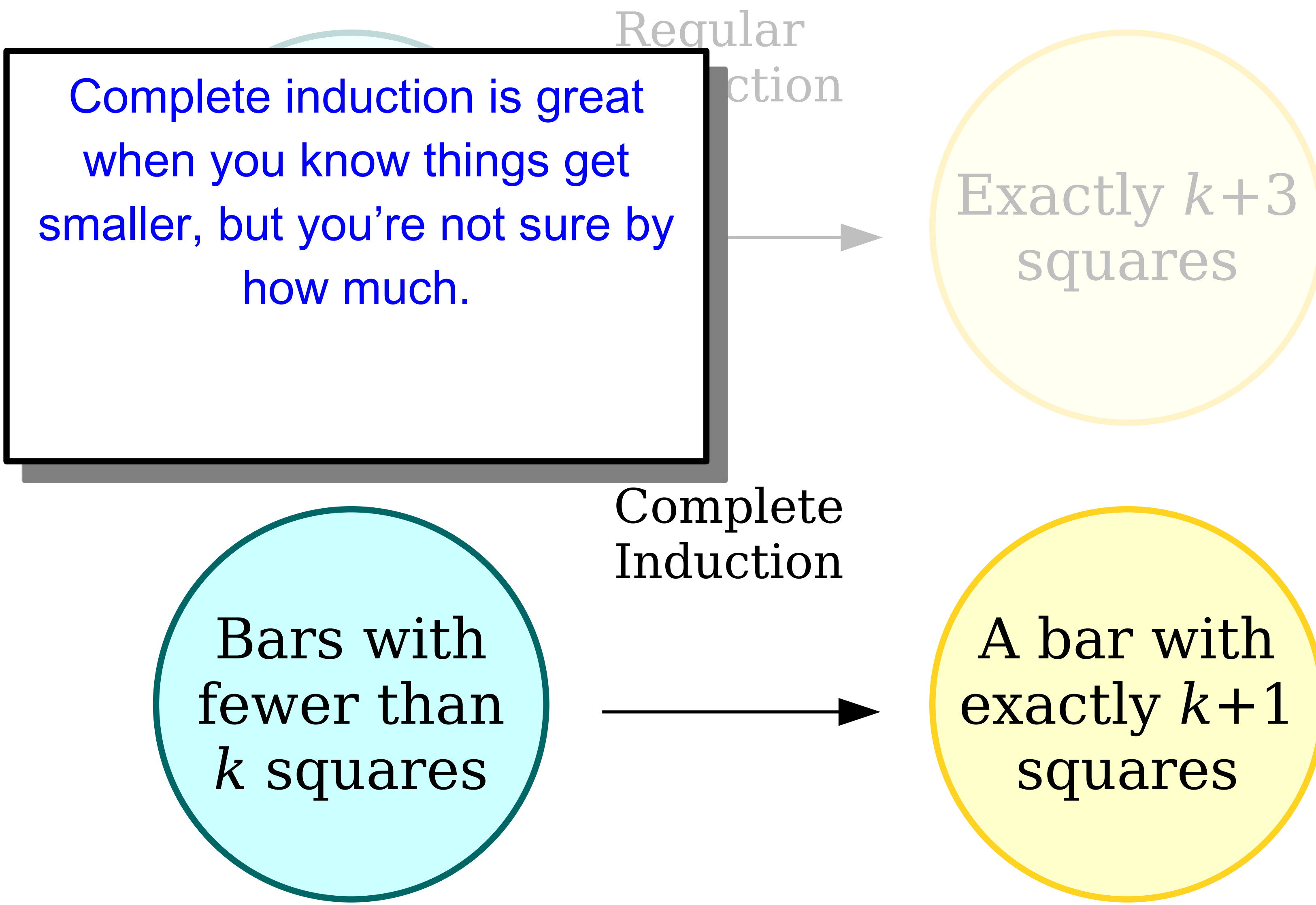

An Important Milestone

Recap: ***Discrete Mathematics***

- The past four weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Formal Proofs	Mathematical Logic
Set Theory	

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Three Questions

- What is something you know now that, at the start of the quarter, you knew you didn’t know?
- What is something you know now that, at the start of the quarter, you didn’t know that you didn’t know?
- What is something you don’t know that, at the start of the quarter, you didn’t know that you didn’t know?

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.